RESEARCH ANNOUNCEMENTS

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PRINCIPAL FUNCTIONS FOR ELLIPTIC SYSTEMS OF DIFFERENTIAL EQUATIONS[1]

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Introduction. Let A be an elliptic system of linear differential operators on an open set V of R^n (or, more generally, an elliptic differential operator in a vector bundle B over a manifold V). If V_1 is an open subset of V, the principal function problem for A is intuitively the following: Given a solution s of $As=0$ in V_1, to find a solution p of $Ap=0$ on all of V such that, on V_1, $u=p-s$ is a "nice" solution of $Au=0$, i.e., u satisfies prescribed boundedness and boundary conditions.

For the case of a single self-adjoint second-order elliptic operator, principal functions were introduced by L. Sario and studied systematically by him and his collaborators in [1]–[6] by making strong use of the maximum principle and the Harnack inequality. It is our object in the present paper to indicate an extremely direct and simple proof of the existence of principal functions for the general class of linear elliptic systems of differential operators.

1. Let V_1 be an open subset of V. We consider r-vector functions $u=(u_1, \cdots, u_r)$ on V; let $x=(x_1, \cdots, x_n)$ be the general point of V, and let $C_c^\infty(V)$ be the family of infinitely differentiable functions u with compact support in V. We shall consider elliptic systems A of the form

$$Au = \sum_{|\alpha| \leq m} A_\alpha(x) D^\alpha,$$

where for each n-tuple $\alpha=(\alpha_1, \cdots, \alpha_n)$ of non-negative integers, D^α is the elementary differential operator $\prod_{j=1}^n (\partial/\partial x_j)^{\alpha_j}$ and A_α is an $(r\times r)$-matrix function on V. For simplicity, we assume that A_α is infinitely differentiable to avoid complication of statement though all statements are valid under very mild regularity conditions.

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DEFINITION 1. *By a boundary-value problem for A on V we mean a class of functions F from $C^\infty(V)$ with F containing $C_c^\infty(V)$.*

DEFINITION 2. *By a normal operator L for A on V_1 with respect to the boundary-value problem given by F, we mean a linear operator defined on $C^\infty(\overline{V}_1)$, where $\overline{V}_1$ is the closure of V_1 in V, with Lv a solution of $Au=0$ on V_1 and such that for each u in F with $Au=0$ on F_1,*

$$L(u \mid _{V_1}) = u \mid _{V_1}.$$

DEFINITION 3. *The boundary-value problem for A on V given by F is said to be solvable if for each f in $C_c^\infty(V)$, there exists u in F such that $Au=f$ in V.*

THE PRINCIPAL FUNCTION PROBLEM. *Let A be an elliptic system on V, V_1 an open subset of V with $V-V_1$ compact. Suppose that F is a boundary-value problem on V, L a normal operator for A on V_1 with respect to F. Then, given a solution s in $C^\infty(\overline{V}_1)$ of the equation $As=0$ in V_1, we seek a function p in $C^\infty(V)$ such that $Ap=0$ in V, while*

$$p \mid _{V_1} - s = L(p \mid _{V_1} - s).$$

THEOREM. *If F defines a solvable boundary-value problem for A on V, then there exists a principal function p for every s.*

PROOF OF THE THEOREM. By hypothesis, s lies in $C^\infty(\overline{V}_1)$, where $\overline{V}_1$ is the closure of V_1 in V. Hence, it may be extended to an element s_0 of $C^\infty(V)$. Since $As=0$ on V_1, As_0 lies in $C_c^\infty(V)$. We consider the boundary-value problem

$$Au = -As_0, \qquad u \in F.$$

By hypothesis, there exists a solution u and $Au=0$ in V_1. Let

$$p = s_0 + u.$$

Then p lies in $C^\infty(V)$, $Ap=As_0-As_0=0$, while

$$\begin{aligned} p \mid _{V_1} - s &= (p - s_0) \mid _{V_1} = u \mid _{V_1} \\ &= L(u \mid _{V_1}) = L(p \mid _{V_1} - s), \end{aligned} \qquad \text{q.e.d.}$$

EXAMPLE. Let A be given in the form

$$Au = \sum_{|\alpha|,|\beta| \leq m} D^\beta(A_{\alpha\beta}(x) D^\alpha u)$$

and consider the Dirichlet form $a(u, v)$ defined by

$$a(u, v) = \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\beta|}(A_{\alpha\beta}(x) D^\alpha u, D^\beta v),$$

with (u, v) the L^2-inner product. If we suppose that $a(u, v)$ is Hermitian and positive on $C_c^\infty(V)$, it turns $C_c^\infty(V)$ into a pre-Hilbert space. Let H be the completion of this pre-Hilbert space and suppose that H can be realized as a space of distributions. Then by the standard orthogonal projection argument, the boundary-value problem defined by $F=H\cap C^\infty(V)$ is solvable for A on V.

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