INVARIANCE OF THE ESSENTIAL SPECTRUM[1]

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Let A be a densely defined linear operator in a Banach space X. Wolf [4] defines the *essential spectrum* of A as the complement of Φ_A, the set of those complex λ for which $A-\lambda$ is closed and

(a) $\alpha(A-\lambda)$, the multiplicity of λ, is finite,

(b) $R(A-\lambda)$, the range of $A-\lambda$, is closed,

(c) $\beta(A-\lambda)$, the codimension of $R(A-\lambda)$, is finite.

This set, which we denote by $\sigma_{ew}(A)$, has the desirable property of being invariant under arbitrary compact perturbations. The largest subset of the spectrum having this property will be denoted by $\sigma_{em}(A)$. It is obtained by adding to $\sigma_{ew}(A)$ those points of Φ_A for which $i(A-\lambda)\equiv\alpha(A-\lambda)-\beta(A-\lambda)\neq 0$.

Browder [2] has given a different definition. It is equivalent to adding to $\sigma_{em}(A)$ those spectral points which are not isolated. We denote the essential spectrum according to this definition by $\sigma_{eb}(A)$. It has the advantage of excluding only isolated points of the spectrum, but is more delicate with respect to perturbations.

In this paper we give sufficient conditions for the invariance of each of the sets $\sigma_{ew}(A)$, $\sigma_{em}(A)$, $\sigma_{eb}(A)$ under perturbations.

Let B be a linear operator in X with $D(A)\subseteq D(B)$. It is called A-bounded if $\|Bx\|\leqq\text{const.}(\|x\|+\|Ax\|)$ for all $x\in D(A)$. It is called A-compact if $\|x_n\|+\|Ax_n\|\leqq\text{const.}$ for $\{x_n\}\subseteq D(A)$ implies that $\{Bx_n\}$ has a convergent subsequence. We shall call B *A-closed* if $x_n\to x$, $Ax_n\to y$, $Bx_n\to z$ implies that $x\in D(B)$ and $Bx=z$. It will be called *A-closable* if $x_n\to 0$, $Ax_n\to 0$, $Bx_n\to z$ implies $z=0$.

THEOREM 1. *If B is A-compact, then*

$$\sigma_{ew}(A+B)\subseteq\sigma_{ew}(A), \tag{1}$$

$$\sigma_{em}(A+B)\subseteq\sigma_{em}(A). \tag{2}$$

If, in addition, one of the following holds

(a) *A is closable,*

(b) *B is $(A+B)$-closable,*

(c) *B is A-closed,*

then

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$$\sigma_{ew}(A+B)=\sigma_{ew}(A), \tag{3}$$

$$\sigma_{em}(A+B)=\sigma_{em}(A). \tag{4}$$

COROLLARY 1. *If B is A-compact and Φ_A is not empty, then* (3) *and* (4) *hold.*

COROLLARY 2. *If B is compact, then* (3) *and* (4) *hold.*

The operator B will be called *A-pseudo-compact* if

$$\|x_n\|+\|Ax_n\|+\|Bx_n\| \leqq \text{const.}, \qquad \{x_n\}\subseteq D(A), \tag{5}$$

implies that $\{Bx_n\}$ has a convergent subsequence. It will be called *A^2-pseudo-compact* if

$$\|x_n\|+\|Ax_n\|+\|Bx_n\|+\|A^2x_n\|+\|BAx_n\| \leqq \text{const.}$$

for $\{x_n\}\subseteq D(A^2)$ implies the same.

THEOREM 2. *If B is A-closable and A-pseudo-compact, then* (1) *and* (2) *hold. If, in addition, it is A-closed, then* (3) *and* (4) *hold.*

THEOREM 3. *If B is A-pseudo-compact and neither Φ_A nor Φ_{A+B} is empty, then* (3) *and* (4) *hold.*

THEOREM 4. *Assume that $\lambda\in\rho(A)\cap\Phi_{A+B}$. If* (5) *implies that $\{(A-\lambda)^{-1}Bx_n\}$ has a convergent subsequence, then* (3) *holds. If, in addition, $i(A+B-\lambda)=0$, then* (4) *holds. The same is true when $(A-\lambda)^{-1}B$ is replaced by $B(A-\lambda)^{-1}$.*

COROLLARY 3. *If $\lambda\in\rho(A)\cap\Phi_{A+B}$, $i(A+B-\lambda)=0$ and either $(A-\lambda)^{-1}B$ or $B(A-\lambda)^{-1}$ is A-compact, then* (3) *and* (4) *hold.*

COROLLARY 4. *If $\lambda\in\rho(A)\cap\rho(A+B)$ and $(A-\lambda)^{-1}-(A+B-\lambda)^{-1}$ is a compact operator, then* (3) *and* (4) *hold.*

Corollary 4 exhibits a device employed by Birman [1] and Wolf [4].

THEOREM 5. *Assume that $\lambda\in\Phi_A\cap\Phi_{A+B}$. If B is A^2-pseudo-compact, then* (3) *holds. If, in addition, $i(A-\lambda)=i(A+B-\lambda)$, then* (4) *holds as well.*

THEOREM 6. *Suppose that B is A-pseudo-compact and that the complement $C\sigma_{em}(A)$ of $\sigma_{em}(A)$ is connected. If neither $\rho(A)$ nor $\rho(A+B)$ is empty, then*

$$\sigma_{eb}(A+B)=\sigma_{eb}(A). \tag{6}$$

THEOREM 7. *Assume that* $\lambda \in \rho(A) \cap \Phi_{A+B}$, *that* $C\sigma_{em}(A)$ *is connected, and that* $\rho(A+B)$ *is not empty. If* (5) *implies that* $\{(A-\lambda)^{-1}Bx_n\}$ *has a convergent subsequence, then* (6) *holds. The same is true if* $(A-\lambda)^{-1}B$ *is replaced by* $B(A-\lambda)^{-1}$.

COROLLARY 5. *If* $C\sigma_{em}(A)$ *is connected*, $\lambda \in \rho(A) \cap \rho(A+B)$, *and the operator* $(A-\lambda)^{-1}-(A+B-\lambda)^{-1}$ *is compact, then* (6) *holds.*

THEOREM 8. *Assume that* $\lambda \in \Phi_A \cap \Phi_{A+B}$, $i(A-\lambda)=i(A+B-\lambda)$, *and that* $\rho(A)$ *and* $\rho(A+B)$ *are not empty. If* $C\sigma_{em}(A)$ *is connected and* B *is* A^2*-pseudo-compact, then* (6) *holds.*

THEOREM 9. *Let* Ω *be an open connected set containing no points of* $\sigma_{eb}(A)$. *There is an* $\epsilon > 0$ *such that*

$$\|Bx\| \leqq \epsilon(\|x\| + \|Ax\|), \qquad x \in D(A),$$

implies for B A^2*-pseudo-compact that* $\sigma_{eb}(A+B) \cap \Omega$ *is empty.*

THEOREM 10. *If* $\sigma_{eb}(A)=\sigma_{em}(A)$ *and* B *is both* A*-compact and* A*-closed, then there is at most a denumerable set* S *of complex numbers* μ *for which*

$$\sigma_{eb}(A + \mu B) = \sigma_{eb}(A)$$

fails to hold. If $C\sigma_{eb}(A)$ *has only a finite number of components, then* S *is discrete.*

The following generalization of a result of Taylor [3] is useful in the proofs.

LEMMA. *If* Φ_A *is not empty, then for any polynomial* $p(\eta)$, $p(A)$ *is a closed operator.*

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