DIRECT FACTORS OF \((AL)\)-SPACES

BY DAVID W. DEAN

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Let \(E\) be a closed sublattice of the \((AL)\)-space \(L\) [1, pp. 107–110]. The purpose of this note is to prove that there is a projection of \(L\) onto \(E\) having norm one. In particular then \(E\) is a direct factor of \(L\). To show this we prove auxiliary theorems that the conjugate space \(E'\) of \(E\) may be “lifted” to \(L'\) (see Theorem 2 below) and that there is a projection of \(E'' = (E')'\) onto \(q(E)\), the natural embedding of \(E\) in \(E''\), whose norm is one.

The space \(L'\) is isometric and lattice isomorphic to a space \(C(H)\) of functions continuous on a compact, extremally disconnected Hausdorff space \(H\) [4, Theorems 6.3, 6.9, Corollary 6.2]. Then \(L''\) is isometric and lattice isomorphic to the space \(R(H)\) of regular measures on \(H\) [3, p. 265]. If \(x' \in L', x'' \in L''\) correspond to \(f \in C(H), \nu \in R(H)\), then \(x''(x') = \int_H f \, d\nu = (\text{def. } \nu(f))\). If \(\nu \in R(H)\) and \(\nu(N) = 0\) for each nowhere dense set \(N\) then \(\nu\) is a normal measure. The support \(A\), [2, pp. 2, 8, Proposition 3] of such a measure is both open and closed. Let \(N(H)\) denote the subspace of normal measures.

**Theorem 1.** The representation of \(L''\) as \(R(H)\) maps \(q(L)\) onto the space \(N(H)\) of normal measures on \(H\). Moreover \(\bigcup \{A, \nu \in N(H)\}\) is dense in \(H\) (so that \(H\) is hyperstonean [2]).

**Proof.** Let \(\nu \geq 0\) correspond to \(qx\) for \(x\) in \(L\). Let \(N\) be a closed nowhere dense set. We prove first that \(\nu(N) = 0\). Let \(F\) be the subset of functions \(f\) in \(C(H)\) for which \(|f| = 1, f \geq 0, f(h) = 1\) if \(h \in N\). Then \(F\) is directed by \(\geq\). This directed set then converges at each such \(\nu\) to \(\inf \{\nu(f) | f \in F\}\). Thus \(F\) converges on the representation of \(L\) in \(R(H)\). The directed set of \(x'\) in \(L'\) corresponding to \(F\) then converges pointwise on \(L\) to an element \(y'\) in \(L'\). If \(y'\) corresponds to \(g\) in \(C(H)\) we have \(\nu(g) = \inf \{\nu(f) | f \in F\}\) and clearly \(g = \inf \{f \in F\}\). Since \(N\) is nowhere dense, \(g = 0\). Thus \(\inf \{\nu(f) | f \in F\} = 0\) so that \(\nu(N) = 0\). Thus \(\nu\) is a normal measure.

To prove the second part let \(A\) be open and closed in \(H\). For some \(\nu > 0\) corresponding to \(qx, x \in L\), we have \(\nu(\chi_A) > 0\), where \(\chi_A\) is the characteristic function of \(A\). Thus \(A\) meets the support of \(\nu\). Hence \(H\) is hyperstonean. The theorem follows immediately from a result of Dixmier [2, p. 21, the corollary and its proof].

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THEOREM 2. Let $i$ be the inclusion mapping $E \rightarrow L$ and let $i': L' \rightarrow E'$ denote the conjugate mapping. There is a positive isometry $T: E' \rightarrow L'$ such that $i'T$ is the identity on $E'$. Thus $Tx'$ is an extension of $x'$ to $L$ for each $x'$ in $E'$.

PROOF. The space $E$ is itself an $(AL)$-space and so it is isometric and lattice isomorphic to a space $N(K)$, with conjugate $C(K)$, and second conjugate $R(K)$, as above. We shall then, suppressing the representation mappings and their inverses, write $N(K) \rightarrow iN(H)$, $C(H) \rightarrow i'C(K)$, and seek a positive isometry $T: C(K) \rightarrow C(H)$ with the property that $i'T$ is the identity on $C(K)$.

Let $\chi_A$ denote the characteristic function of the set $A$. Let $\alpha$ be the collection of open and closed subsets of $H$ such that $i'\chi_A = 0$ and let $G = \text{Cl}(U \{A \in \alpha\})$. Then $G$ is open and closed and we now show that $i'\chi_A = 0$. It is enough to show $\nu(G) = 0$ if $\nu \geq 0$ is in $i(N(K))$ since $i(N(K))$ is a sublattice of $N(H)$. Since $\nu$ is normal, $\nu(G) = \nu(U \{A \in \alpha\})$ as $G = \bigcup \{A \in \alpha\}$ is nowhere dense. If $C \subseteq \bigcup \{A \in \alpha\}$ is closed, a finite number of $A$'s in $\alpha$ cover $C$, so that $\nu(C) = 0$. Since $\nu$ is regular, $\nu(U \{A \in \alpha\}) = 0$.

Now let $e$ be an extreme point of the unit ball of $C(K)$ (so that $e$ takes only the values 1 and $-1$). As a functional on $N(K)$ $e$ has an extension to $C(H)$ which is an extreme point of the unit ball of $C(H)$. This follows since the set of norm one extensions of $e$ is a compact convex set in the $w^*$-topology of $C(H)$. This set then has an extreme point and such a point is also an extreme point of the unit ball of $C(H)$. Let $f$ agree with such an extension off $G$ and have value 0 on $G$. Then $f$ is an extension of $e$ to $C(H)$ and takes only the values 1 and $-1$ off $G$. Then $f$ has the following properties. (a) $i^*f^* = e^+$ (so $i^*f^- = e^-$), (b) $f$ is unique. To show (a) note that $\chi_K \geq i^*f^* \geq 0$, $\chi_K \geq i^*f^- \geq 0$ and $i^*f = i^*(f^* - f^-) = e$ ($i'$ is a positive norm one mapping). Thus if $e(k) = 1$ then $i^*f^*(k) = 1$ and if $e(k) = -1$ then $i^*f^-(k) = 1$. Hence (a). To show (b) let $g$ be another such $f$. By (a) $i^*g^* = e^+$. Let $A' = \{h | g(h) = 1, f(h) = -1\}$. Then $A'$ is open and closed and $g^+ \geq \chi_{A'} \geq 0$. Thus $e^+ \geq i'\chi_{A'} \geq 0$. However $f^- \geq f^* - \chi_{A'} \geq 0$ so that $e^+ \geq i'\chi_{A'} \geq 0$. Thus $i'\chi_{A'}(k) = 0$ if $e^+(k) = 0$ or $e^-(k) = 0$, or $\nu(C) = 0$. It follows that $A' \subseteq G$ so that $A' = \emptyset$. Interchanging $f$ and $g$ in this argument yields $f = g$.

From these calculations one has that, given an open and closed set $A \subseteq K$, there is a unique open and closed set $A' \subseteq H - G$ such that $i'\chi_{A'} = \chi_A$ (let $e = \chi_A - \chi_{K - A}$ and select $\chi_{A'} = f^+$ as above). If $A$ and $B$ are open and closed and if $A \cap B = \emptyset$, it is easy to see that $(A \cup B)' = A' \cup B'$ and that $A' \cap B' = \emptyset$. If $A \cap B \neq \emptyset$ write $(A \cup B)'
\[ ((A-B) \cup (A \cap B) \cup (B-A))' = (A-B)' \cup (A \cap B)' \cup (B-A)' \]

\[ = [(A-B)' \cup (A \cap B)'] \cup [(A \cap B)' \cup (B-A)'] = A' \cup B'. \]

Noting that \( K' = H-G \) and that \( (K-A)' = H-(A' \cup G) \) one gets, by considering complements, that \( (A \cap B)' = A' \cap B' \) for all open and closed sets \( A, B \subseteq K \). Thus \( ' \) preserves the ring operations of the ring of open and closed subsets of \( K \).

Now let \( S \) be the submanifold of functions in \( C(K) \) assuming only finitely many values. Define \( T: S \to C(H) \) by \( T(\sum \alpha_i \chi_{A_i}) = \sum \alpha_i \chi_{A_i} \) for \( s = \sum \alpha_i \chi_{A_i} \in S \). Since \( ' \) preserves the ring operations it easily follows that \( T \) is linear and positive and that \( ||Ts|| = ||s|| \) for all \( s \in S \). Now \( S \) is dense in \( C(K) \) as follows. If \( \varepsilon > 0 \) the set \( \{k \in K : ||f|| + n \varepsilon < f(k) < ||f|| + (n+1) \varepsilon, n = 0, 1, 2, \ldots \} \) has open and closed closure \( A_n \). The set \( B_n \) of \( k \) such that \( f(k) = ||f|| + n \varepsilon \) and \( k \notin A_{n-1} \cup A_n \) is also open and closed. At most a finite number of \( A_n \), \( B_n \) are nonempty so \( || \sum \chi_{A_n} + \sum \chi_{B_n} - f || \leq \varepsilon \) if \( M \) is large. Thus \( T \) has an extension to all of \( C(K) \) (also denoted by \( T \)) which is positive and an isometry.

Let \( q \) be the natural embedding of \( E \) in \( E' \) or of \( L \) in \( L' \). Thus \( q(f) = f \) for all \( f \) in \( E' \), \( \nu \) in \( E \) (or \( f \) in \( L' \), \( \nu \) in \( L \)).

**Theorem 3.** There is a norm one projection from \( E' \) onto \( q(E) \).

Suppose for the moment this theorem has been proved. Let \( T \) be the isometry \( E' \to L' \) promised in Theorem 2. The inclusion mapping \( i \) is suppressed in the following argument. Then \( T': L'' \to E'' \) and for \( x \) in \( E \) one has that \( T'qx = qx \) since \( T'q(x') = qx(T'x') = Tx'(x) = x'(x) = qx(x') \) for every \( x' \) in \( E' \). Thus \( T'q(L) \supseteq q(E) \) in \( E'' \). By Theorem 3 there is a projection \( P \) of \( E'' \) onto \( q(E) \) such that \( ||P|| = 1 \). Then \( P \) restricted to \( T'q(L) \) is a projection of \( T'q(L) \) onto \( q(E) \). Finally \( Q = q^{-1}PT'q \) is a projection of \( L \) onto \( E \) having norm one since clearly \( ||Q|| = 1 \), \( Q: L \to E \), and \( Q \) is the identity on \( E \). Thus one has

**Theorem 4.** If \( E \) is a closed sublattice of the \((AL)\)-space \( L \) there is a projection \( Q \) of \( L \) onto \( E \) such that \( ||Q|| = 1 \).

**Proof of Theorem 3.** Identify \( E'' \) with the space \( R(K) \) so that \( q(E) \) is identified with \( N(K) \). It is sufficient to show there is a norm one projection of \( R(K) \) onto \( N(K) \). Let \( \mathfrak{N} \) be the set of closed nowhere dense subsets of \( K \). Let \( \nu \geq 0 \) be in \( R(K) \). Define \( \nu_1 \) on an open and closed set \( A \) by \( \nu_1(A) = \sup \{ \nu(N) : N \subseteq A, N \in \mathfrak{N} \} \). Then \( \nu_1 \) is finitely additive on the ring of open and closed sets. If \( \sum \alpha_i \chi_{A_i} \in S \), then
\( \nu_1(\sum_1^n a_j\chi_{A_j}) = \sum_1^n a_j\nu(A_j) \) defines a continuous linear functional on \( S \) whose extension to \( C(K) \) yields an element \( \nu_2 \leq \nu \) of \( R(K) \). We will show that \( \nu(N) = \nu_2(N) \) for all \( N \in \mathfrak{N} \). Choose an open set \( B \supset N \) such that \( \nu_2(B - N) < \varepsilon \). Choose \( f \) in \( C(H) \) such that \( ||f|| = 1, f(k) = 1 \) if \( k \in K-B \) and \( f(k) = 0 \) if \( k \in N \). \( A = \text{Cl}(\{ k \mid f(k) < \frac{1}{2} \}) \) is an open and closed set for which \( \nu_2(A - N) < \varepsilon \) and \( N \subset A \). Then \( \nu_2(N) \leq \nu(N) \leq \nu_2(A) \leq \nu_2(N) + \varepsilon \) so that \( \nu_2(N) = \nu(N) \). Choose \( f = \nu - \nu_2 \). Clearly \( 0 \leq \nu_2 \leq \nu \) and \( \nu_2 \in N(K) \). If we define \( P(\nu) = \nu_2 \) for \( \nu \geq 0 \) then \( P(a\nu) = aP(\nu) \) if \( a \geq 0 \) and \( P(\nu + \mu) = P(\nu) + P(\mu) \), \( \nu, \mu \geq 0 \). Now any \( \nu \) can be written \( \nu = \mu - \lambda \) for some \( \lambda, \mu \geq 0 \), and we define \( P(\nu) = P(\mu) - P(\lambda) \). If \( \nu = \mu_1 - \mu_1 = \mu_2 - \lambda_2 \) in this way then \( \mu_1 + \lambda_2 = \mu_2 + \lambda_1 \) so \( P(\mu_1) + P(\lambda_2) = P(\mu_2) + P(\lambda_1) \) or \( P(\mu_1) - P(\lambda_1) = P(\mu_2) - P(\lambda_2) \) and thus \( P \) is well defined. Moreover \( P \) is clearly linear and \( ||P|| = 1 \). Q.E.D.

**Bibliography**


**Ohio State University**