DIFFERENTIABLE FUNCTIONS ON CERTAIN BANACH SPACES

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The main result in this note, Theorem 2, can be thought of as a very strong maximum modulus type theorem. For example, let D be a bounded connected open set in $C(0, 1)$, and let f: $\mathrm{Cl}D \to R^n$ be continuous and differentiable in D. Then f is determined by its values on the boundary of D. More exactly, $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$. More generally, if F is any Banach space and f: $\mathrm{Cl}D \to F$ is completely continuous and differentiable in D, then $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$. Note that these results are false if $C(0, 1)$ is replaced by a Hilbert space.

THEOREM 1. *Let D be a connected bounded open set in l^p where p is not an even integer. Assume f is a real-valued function, continuous on* $\mathrm{Cl}D$ *and n-times differentiable in D with $n \geqq p$. Then* $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$.

This generalizes a result proved in 1954 by Kurzweil [1]. Kurzweil assumed that f was n-times continuously differentiable, that D was a ball $B(x_0, r)$, and showed that $\inf\{|f(x) - f(x_0)| : \|x - x_0\| = r\} = 0$.

COROLLARY 1. *Let f be an n-times differentiable function on l^p, where $n \geqq p$, and p is not an even integer. If f has its support in a bounded set, then f is identically zero.*

In particular, it follows that, for $n \geqq p$, C^n partitions of unity do not exist whenever p is not an even integer. This partially settles a question raised in Lang [2]. It should be noted, however, that this is implied by Kurzweil's result.

COROLLARY 2. *Let E be a Banach space containing a subspace equivalent to l^1. Assume D is a connected bounded open set in E, and that f is a real-valued function continuous on* $\mathrm{Cl}D$ *and differentiable in D. Then* $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$.

$C(0, 1)$ and $L^1(0, 1)$ are examples of spaces where Corollary 2 holds. More generally, any separable Banach space with an unconditional basis and nonseparable dual contains a subspace equivalent to l^1. It may be that any separable Banach space with a nonseparable dual has a subspace equivalent to l^1. Corollary 2 generalizes an unpublished result of Edward Nelson who showed that, in $C(0, 1)$, differentiable functions with bounded support are identically zero.

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THEOREM 2. *Let E be a Banach space containing a subspace equivalent to l^1, let F be any Banach space, and let D be a bounded open connected set in E. Assume f: $\mathrm{Cl}D \to F$ is continuous, and that $f'(x)$ exists and is a completely continuous mapping for all $x \in D$. Then $f(\mathrm{Cl}D) \subset \mathrm{Cl}f(\partial D)$.*

COROLLARY 1. *Let E and F be as in the theorem and let T: $E \to F$ be completely continuous and differentiable. Then $T(\mathrm{Cl}D) \subset \mathrm{Cl}T(\partial D)$ for any bounded connected open set $D \subset E$.*

This follows from the fact that if T: $E \to F$ is completely continuous and $T'(x)$ exists, then $T'(x)$ is a completely continuous linear mapping.

Letting F be the reals gives the following "sup principle".

COROLLARY 2. *Let E and D be as in the theorem, and let f be a real-valued function continuous on $\mathrm{Cl}D$ and differentiable in D. Then* $\sup_{\mathrm{Cl}D} f(x) = \sup_{\partial D} f(x)$.

Note that $f(x) = 1 - \|x\|^2$ shows that E cannot be replaced by a Hilbert space.

COROLLARY 3. *Let M be a differentiable manifold modelled on E where E contains a subspace equivalent to l^1, and let N be any differentiable manifold. Suppose f: $M \to N$ is differentiable and, for each x, $f'(x)$: $T_x(M) \to T_{f(x)}(N)$ is a completely continuous mapping. Let (U, g) be a chart where $gU \subset E$ is bounded, open, and connected. Then $f(\mathrm{Cl}\,U) \subset \mathrm{Cl}f(\partial U)$.*

It is well known that if p is an even integer, the norm on l^p is C^∞, and if p is not even the norm is C^q, where q is the greatest integer strictly less than p. The argument in Lang [2] then shows that C^∞- and C^q-approximation holds in these respective spaces. It follows from Theorem 1 that for p not even, $(q+1)$-differentiable approximation does not hold. Restrepo [3] showed that a Banach space has an equivalent C^1-norm if and only if its dual space is separable. It follows that C^1-approximation then holds for such spaces. It follows from Theorem 2 that if E is a Banach space containing a subspace equivalent to l^1, then not even differentiable approximation holds. In the following we show that C^∞-approximation holds for c_0. Restrepo's result shows that c_0 has an equivalent C^1-norm, and it is natural to ask if c_0 has an equivalent C^∞-norm. However, we do not even know if c_0 has an equivalent C^2-norm. This result has also been observed by Edward Nelson.

REMARK. C^∞-approximation holds in c_0.

Simply let g: $R \rightarrow R$ be *a* C^∞ function satisfying $g(t)=1$ for $|t| \leqq 1/2$, $g(t)=0$ for $|t| \geqq 1$, and $0<g(t) \leqq 1$ for $(1/2)<|t|<1$. Let $x=(x_1, x_2, \cdots) \in c_0$ and define $f(x)=\prod_{i=1}^{\infty} g(x_i)$. Then f is a C^∞ function nonzero in the open unit ball, and zero off it. The argument is then completed as in Lang [**2**].

Complete details, extensions, and applications of the results in this note will be published elsewhere.

BIBLIOGRAPHY

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3. G. Restrepo, *Differentiable norms in Banach spaces*, Bull. Amer. Math. Soc. **70** (1964), 413–414.

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