WEIGHTED ENTIRE FUNCTIONS

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Communicated by R. Boas, November 9, 1964

The aim of this note is to announce some results concerning the weighted entire function \( z^\alpha f(z) \), where \( \alpha \geq 0 \) and \( f(z) \) is an entire function of exponential type. All these results are known to be true when \( \alpha = 0 \). Boas \[2\] proved that if \( f(z) \) is an entire function of exponential type such that \( f(x) \in L^p(-\infty, \infty) \), then \( f'(x) \in L^p(-\infty, \infty) \). Plancherel and Pólya \[6\] showed that if \( f(z) \) is an entire function of exponential type such that \( f(x) \in L^p(-\infty, \infty) \), then \( \sum_{n=-\infty}^{\infty} |f(n)|^p < \infty \); if the type of \( f(z) \) is less than \( \pi \), then the converse is also true. Harvey \[4\] proved a number of results concerning the \( p \)-th mean values of an entire function of exponential type. In this note we generalize all these results to weighted entire functions. These results will appear with proofs soon.

Let \( z = x + iy \), where \( x \) and \( y \) are real, denote the complex variable. Suppose \( p > 0 \), \( \alpha \geq 0 \) and \( f(z) \) is an entire function of exponential type. We set, for each fixed real number \( \alpha \),

\[
M_T^{\alpha}[f(x + a + iy)] = \frac{1}{2T} \int_{-T}^{T} \left| x^\alpha f(x + a + iy) \right|^p dx.
\]

If \( n \) is a positive integer, we set, for each fixed integer \( m \),

\[
N_n^{\alpha}[f(x + m)] = (2n + 1)^{-1} \sum_{k=-n}^{n} \left| k^\alpha f(k + m) \right|^p.
\]

We define

\[
M^{\alpha}[f(x + iy)] = \limsup_{T \to \infty} M_T^{\alpha}[f(x + iy)]
\]

and

\[
N^{\alpha}[f(x)] = \limsup_{n \to \infty} N_n^{\alpha}[f(x)].
\]

Here \( M^{\alpha}[f(x)] \) and \( M^{\alpha}[f(x + iy)] \) are the weighted \( p \)-th mean of \( f(z) \) along the real axis and along a line parallel to the real axis, re-
spectively, with weight $x^a$, whereas $N^{p,a}[f(x)]$ is the weighted mean at the integers.

Our principal result is

**Theorem.** Let $\alpha > 0$, $p > 0$ and let $f(x)$ be an entire function of exponential type $\tau$. Then

1. $x^\alpha f(x) \in L^p(-\infty, \infty)$ implies $\sum_{n=-\infty}^{\infty} |n^\alpha f(n)|^p < \infty$;
2. $x^\alpha f(x) \in L^p(-\infty, \infty)$ implies
   \[ \int_{-\infty}^{\infty} |x^\alpha f(x + iy)|^p \, dx \leq e^{\pi |y|} \int_{-\infty}^{\infty} |x^\alpha f(x)|^p \, dx; \]
3. $x^\alpha f(x) \in L^p(-\infty, \infty)$ implies $x^\alpha f'(x) \in L^p(-\infty, \infty)$;
4. $M^{p,a}[f(x+\alpha+iy)] = M^{p,a}[f(x+iy)]$ for each fixed real number $\alpha$;
5. $N^{p,a}[f(x+m)] = N^{p,a}[f(x)]$ for each fixed integer $m$;
6. if $M^{p,a}[f(x)] = A < \infty$, then
   \[ x^\alpha f(x) = O(\sqrt[1/p]{x}) \quad \text{as} \quad x \to \infty; \]
7. if $N^{p,a}[f(x)] = A < \infty$, and $n$ is an integer, then
   \[ n^\alpha f(n) = O(n^{1/p}) \quad \text{as} \quad n \to \infty; \]
8. $M^{p,a}[f(x+iy)] \leq e^{\pi |y|} M^{p,a}[f(x)]$;
9. $M^{p,a}[f'(x)] \leq \frac{(p+2)e^{\pi\delta}{2}^{p+1}}{p \pi \delta^{p+1}} (e^{\pi \delta} - 1) M^{p,a}[f(x)]$,

where $\delta$ is an arbitrary positive number;

10. $M^{p,a}[f(x)] < \infty$ implies $M^{p,a}[f(x+iy)]$ is a continuous function of $y$ provided $p > 1$;
11. there exists a constant $B > 0$ which depends on $p$ and $\tau$ only such that
   \[ N^{p,a}[f(x)] \leq BM^{p,a}[f(x)]. \]

Moreover, if $\tau < \pi$,
12. converse of (1) is true;
13. converse of (11) is true; more precisely, there exists a constant $C$ which depends on $p$, $\tau$ and $\alpha$ such that
   \[ M^{p,a}[f(x)] \leq CN^{p,a}[f(x)]. \]

The results (4)–(11) and (13) are generalizations of those of Harvey [4] and the methods and techniques are analogous to his.

**Note.** (1) and (12) can be easily proved by the interpolation formulae and the techniques used in the proof of (11) and (13). These
results are also contained in a theorem of Korevaar [5]. If the type of $f(z)$ is $\pi$, that the conclusions of (12) and (13) may not be true can easily be seen by considering the case when $f(z) = \sin \pi z$ and $\alpha = 1$.

**REFERENCES**


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