

CO-IMMUNE RETRACEABLE SETS

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Our notation and terminology basically follows that found in [2], except with regard to notation for unions and intersections; in a few instances we cite other references for special terms.

The following two propositions are established by a fairly straightforward moveable-markers technique; either proof is only a minor variation on the other.¹

PROPOSITION A. *Let α be an infinite recursively enumerable set. Then there is a countably infinite collection Γ of retraceable sets γ_i such that (i) $i \neq j \Rightarrow \gamma_i \cap \gamma_j = \emptyset$, (ii) each γ_i is the unique infinite set retraced by a certain basic general recursive retracing function (for the notions of retracing function and basic retracing function, see [5]), (iii) $\alpha = \bigcup \Gamma$, and (iv) $\alpha - \gamma_i$ is immune for all i .*

PROPOSITION B. *Let α be an infinite recursively enumerable set, and τ an infinite recursive subset of α such that $\alpha - \tau$ is also infinite. Then there is a recursive function f such that, for each i , $f(i)$ indexes a basic, general recursive retracing function which retraces a unique infinite set γ_i , where (i) $i \neq j \Rightarrow \gamma_i \cap \gamma_j = \emptyset$, (ii) each γ_i has exactly one number in common with $\alpha' \cup \tau$, and (iii) $(\alpha - \tau) - \gamma_i$ is immune for all i .*

It was shown by Yates, in [5] (in answer to a question of Dekker and Myhill), that there are basic retracing functions, some of them retracing *unique* infinite sets, which do not retrace any infinite *recursive* set. In each of Yates' examples, all of the sets retraced by such functions have nonimmune complements. The above propositions demonstrate the existence of examples in which an infinite set α is retraced by a basic function and α has immune complement. In any example of this latter type, the function in question *must* retrace a *unique* infinite set, which, of course, cannot be recursive.

We remark that all of the sets γ_i obtained by us in proving Propositions A and B are, owing to the nature of the proofs, *hyperimmune* (for the notion of *hyperimmunity*, see, e.g., [5]). This is closely related to the following general assertion:

¹ We are indebted to Paul Young for a conversation which took place in August, 1963. At that time he made a suggestion which has proved to be susceptible of elaboration into proofs of Propositions A and B.

PROPOSITION C. Let γ be a co-immune retraceable set (i.e., a retraceable set with an immune complement). If γ is not hyperimmune, then (a) no regressing function of γ is potentially recursive (i.e., has a general recursive extension), and (b) if p is a regressing function for γ such that $\rho p \subset \delta p$ (see [2]), then δp is not hypersimple. (For the notion of regressing function of a set, see [2]).

It would be of interest to obtain a co-immune, nonhyperimmune retraceable set; for this would yield an example of an immune regressive set all of whose regressing functions fail to be potentially recursive. (Examples of retraceable sets with no potentially recursive retracing functions, and of nonimmune regressive sets with no potentially recursive regressing functions, are known; see [1] and [4].)

We wish to state a corollary to Propositions A and B which involves the notion of *cohesive set*; some additional terminology is useful in rendering the statement uncluttered. An infinite number set α is called *cohesive* [4] in case there is no recursively enumerable set β for which $\beta \cap \alpha$, $\beta' \cap \alpha$ are both infinite; α is called *supercohesive* in case this holds relative to *regressive* sets β . As was pointed out in [3], every infinite set of numbers has a supercohesive subset, while not all cohesive sets are supercohesive. Let us say that a cohesive set α is of *class* τ , where $1 \leq \tau \leq \aleph_0$, in case there is a collection of τ pairwise disjoint retraceable sets each of which has infinite intersection with α and whose union covers α , but there is no such collection of *more* than τ retraceable sets.

PROPOSITION D. For every τ , $1 \leq \tau \leq \aleph_0$, there exists a cohesive set of class τ . (In the case $\tau = \aleph_0$, it follows from use of Proposition B that the decomposing family of retraceable sets can even be effectively indexed in a certain sense.)

We conclude with a brief discussion of the proofs of Propositions A and B. No attempt is made to define directly the sets γ_i ; rather, a construction is specified for recursive functions which will retrace sets of the required sort. We must make sure that γ_i is co-immune in α (or $\alpha - \tau$); hence, the basic idea of the construction is to tag domain elements of the functions under construction with markers Λ_{ij} which we proceed (subject to various constraints) to *drive downward*, as far as possible, through indices of recursively enumerable subsets of α (or $\alpha - \tau$), with the markers which are attached at a given stage being attached to numbers which have been generated (by the stage in question) in recursively enumerable subsets of α (or $\alpha - \tau$) given by particular indices under scrutiny at that stage. The upshot of the

construction is that all markers eventually stop, and γ_i is the set of numbers k such that the i th constructed function leads down to k from some number m to which a marker Λ_{ij} is attached as its final position; for fixed i , γ_i contains the final positions of all markers Λ_{ij} , $j > 0$, and any infinite recursively enumerable subset of α (or $\alpha - \tau$) contains the final position of a marker Λ_{ij} for *some* j . Making the various retracing functions *total* complicates the basic procedure only slightly.

Added in proof. As to the remark following Proposition C: we have since noticed how, by combining a theorem of D. A. Martin with a result in [4], to prove that, indeed, there exists a retraceable set with no potentially recursive *regressing* function.

REFERENCES

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