ON THE FACIAL STRUCTURE OF CONVEX POLYTOPIES

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A finite family \( C \) of convex polytopes in a Euclidean space shall be called a complex provided

(i) every face of a member of \( C \) is itself a member of \( C \);
(ii) the intersection of any two members of \( C \) is a face of both.

If \( P \) is a \( d \)-polytope (i.e., a \( d \)-dimensional convex polytope) we shall denote by \( B(P) \) the boundary complex of \( P \), i.e., the complex consisting of all faces of \( P \) having dimension \( d-1 \) or less. By \( C(P) \) we shall denote the complex consisting of all the faces of \( P \); thus \( C(P) = B(P) \cup \{P\} \). For a complex \( C \) we define \( \text{set}(C) = \bigcup_{C \in C} C \). For an element \( C \) of a complex \( C \) the closed star [anti-star] of \( C \) (in \( C \)) is the smallest subcomplex of \( C \) containing all the members of \( C \) which contain \( C \) [do not meet \( C \)]. The linked complex of \( C \) in \( C \) is the intersection of the closed star of \( C \) with the anti-star of \( C \).

A complex \( C \) is a refinement of a complex \( K \) provided there exists a homeomorphism \( \phi \) carrying \( \text{set}(C) \) onto \( \text{set}(K) \) such that for every \( K \subseteq K \) there exists a subcomplex \( C_K \) of \( C \) with \( \phi^{-1}(K) = \text{set}(C_K) \).

For example, the complex \( K_1 \) consisting of two triangles with a common edge is a refinement of the complex \( K_2 \) consisting of one triangle; note, however, that the 1-skeleton of \( K_1 \) is not a refinement of the 1-skeleton of \( K_2 \). Let \( \Delta^d \) denote the \( d \)-simplex. The following result is simple but rather useful:

**Theorem 1.** For every \( d \)-polytope \( P \) the complex \( C(P) \) is a refinement of \( C(\Delta^d) \).

**Proof.** The assertion of the theorem is obviously equivalent to the following statement:

**Theorem 1*.** For every \( d \)-polytope \( P \) the complex \( B(P) \) is a refinement of \( B(\Delta^d) \).

We shall prove the theorem in the second formulation, using induction on \( d \). The case \( d = 1 \) being trivial, we may assume \( d \geq 2 \). Let \( V \) be a vertex of \( P \) and let \( H \) be a \((d-1)\)-plane intersecting (in relatively interior points) all the edges of \( P \) incident to \( V \). Then \( P_0 = P \cap H \) is a \((d-1)\)-polytope, and, by the inductive assumption, \( B(P_0) \) is a refinement of \( B(\Delta^{d-1}) \). Let \( S \) denote the closed star of \( V \) in \( B(P) \).
Using radial maps from $V$, it is obvious that the linked complex $L$ of $V$ in $B(P)$ (i.e., the subcomplex of $S$ consisting of all the members of $S$ which do not contain $V$) is a refinement of $B(P_0)$ and thus of $B(\Delta^{d-1})$, while $S$ is a refinement of the closed star $S^*$ of a vertex of $B(\Delta^d)$.

On the other hand, denoting by $A$ the anti-star of $V$ in $B(P)$, set($A$) is homeomorphic to the $(d-1)$-cell $\Delta^{d-1}$ by a homeomorphism carrying set($L$) onto the boundary of $\Delta^{d-1}$. Since $L$ is a refinement of $B(\Delta^{d-1})$, it follows that $A$ is a refinement of $C(\Delta^{d-1})$. Together with the earlier established fact that $S$ is a refinement of $S^*$ this implies (since, on $L$, the two refinements may be chosen to coincide) that $B(P)$ is a refinement of $B(\Delta^d)$, as claimed.

As an immediate consequence of Theorem 1 we obtain the following result [2, Theorem 3]:

**COROLLARY 1.** Every $d$-polyhedral graph contains a refinement of the complete graph with $d+1$ nodes.

**REMARK.** The author is indebted to Dr. Micha Perles for the observation that the proof of Corollary 1, as given in [2], is incomplete, and for indicating how the construction in [2] has to be changed in order to yield a satisfactory proof.

Theorem 1 yields trivially also the following generalization of Corollary 1:

**COROLLARY 2.** For every $k$, $0 \leq k \leq d$, the $k$-skeleton of any $d$-polytope contains a refinement of the $k$-skeleton of $\Delta^d$.

We recall the interesting result of Flores [1] (see also Hurewicz-Wallman [3, p. 63]):

*The n-skeleton of $\Delta^{2n+2}$ is not homeomorphic to a subset of Euclidean $2n$-space.*

Using Schlegel-diagrams, Corollary 2 and Flores' theorem imply:

**THEOREM 2.** The $n$-skeleton of a $(2n+1)$-polytope is not homeomorphic to the $n$-skeleton of a $d$-polytope for $d \geq 2n+2$.

**REFERENCES**


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