A factor is a ring of operators whose center consists only of scalar multiples of the identity. Murray and von Neumann have defined various kinds of factors, calling a continuous factor with finite trace a type $II_1$ factor [3], [4]. Dixmier began the detailed study of maximal abelian subalgebras of type $II_1$ factors. He defined regular, semi-regular but not regular, and singular maximal abelian subalgebras, and showed that at least one of each type exists [2]. His $II_1$ factors turn out to be hyperfinite in algebraic type. The factors we consider are also hyperfinite. In this note we discuss their semi-regular subalgebras, and present an isomorphism invariant which allows us to obtain new existence results.

Let $\mathfrak{A}$ be a hyperfinite factor, $\mathcal{R}$ a maximal abelian subalgebra of $\mathfrak{A}$. For any subring $D$ of $\mathfrak{A}$, $N(D)$ is the ring generated by all unitaries which leave $D$ invariant, and $N^k(D) = N[N^{k-1}(D)]$. In particular, we let $N(\mathcal{R}) = \mathcal{R}$. $\mathcal{R}$ is semi-regular but not regular iff $\mathcal{P}$ is a factor not equal to $\mathfrak{A}$. In [5] we defined an isomorphism invariant for such subalgebras, which we called length. If $\mathcal{R} \subset \mathcal{P} \subset N(\mathcal{P}) \subset \cdots \subset N^L(\mathcal{P}) = \mathfrak{A}$, (when $\mathcal{R} \neq \mathcal{P} \neq N(\mathcal{P}) \neq \cdots \neq N^L(\mathcal{P})$) then $L$ is the length of $\mathcal{R}$. By constructing a semi-regular subalgebra $\mathcal{R}$ of every length $L = 1, 2, 3, \cdots$, we obtained an infinite sequence of subalgebras which could not be pairwise connected by $*$-automorphisms of $\mathfrak{A}$.

Another possible invariant is product type. Suppose $\mathcal{R}$ has length $L$. Then $\mathcal{R}$ is of product type $\alpha$, $0 \leq \alpha \leq L$, iff the following holds: For every $t$, $1 \leq t \leq \alpha$, there exist $S_1$ and $S_2$ in $N^{t-1}(\mathcal{P})^\perp \cap N^t(\mathcal{P})$ such that the product $S_1 S_2 \neq 0$ is in $N^{t-1}(\mathcal{P})^\perp \cap N^t(\mathcal{P})$. But for $s$ such that $\alpha \leq s \leq L$, every $T_1$ and $T_2$ in $N^{s-1}(\mathcal{P})^\perp \cap N^s(\mathcal{P})$ have their product $T_1 T_2$ in $N^{s-1}(\mathcal{P})$. (Taking of orthogonal complements is meaningful, for within a $II_1$ factor, the weak, strong, and Hilbert space (metric) closures of a subalgebra all coincide [4]. The metric topology is based on the norm derived from the scalar product $(A, B) = \text{Tr}(B^*A)$ for $A, B$ in $\mathfrak{A}$.)

Theorem 1. Suppose $\mathcal{R}$ and $\mathcal{R}'$ are semi-regular but not regular subalgebras of $\mathfrak{A}$, and $\mathcal{R}$ has product type $\alpha$, while $\mathcal{R}'$ has product type $\beta$.

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1 This work was done as part of NSF Research Participation for College Teachers, the University of Oklahoma, summer, 1964.
$\alpha', \alpha \neq \alpha'$. Then there does not exist a *-automorphism $\Theta$ of $\mathfrak{A}$ such that $\Theta(R') = R$.

**Proof.** We can assume $\alpha' > \alpha$, so that $\alpha' \geq \alpha + 1$. Letting $t = \alpha + 1$ in the definition of product type, we know we can choose $S_1, S_2$ in $N^a(P') \cap N^{a+1}(P')$ such that $S_1 S_2 = S_3 \neq 0$ is in this set also. Suppose there exists $\Theta$ such that $\Theta(R') = R$. By a standard argument, it follows that $\Theta[N(R')] = N(R)$ or $N(P') = P$, and inductively, $\Theta[N^a(P')] = N^a(P)$ and $\Theta[N^{a+1}(P')] = N^{a+1}(P)$. Let $\Theta(S_i) = T_i$, so that $T_i \in N^{a+1}(P)$ for $i = 1, 2, 3$. Now if $A \in N^a(P')$, then $(S_i, A) = 0 = \text{Tr}(A \ast S_i) = \text{Tr}[\Theta(A \ast S_i)]$ (since the trace function is unique) $= \text{Tr}[\Theta(A) \ast \Theta(S_i)] = (\Theta(S_i), \Theta(A)) = (T_i, \Theta(A)) = 0$. As $A$ ranges over $N^a(P')$, $\Theta(A)$ takes on all values in $N^a(P)$, so we must have $T_i \in N^a(P)$. Thus $T_i$ is in $N^a(P) \cap N^{a+1}(P)$ for $i = 1, 2, 3$.

Now $\alpha + 1 > \alpha$, so letting $s = \alpha + 1$ and considering the product type of $R$, it follows that $T_1 T_2$ is in $N^a(P)$. But $T_1 T_2 = \Theta(S_1) \Theta(S_2) = \Theta(S_1 S_2) = \Theta(S_3) = T_3$. Since $T_3$ is in $N^a(P)$, this leads to a contradiction. ($T_3 \neq 0$ since $S_3 \neq 0$ and $\Theta$ is an automorphism.) Therefore we cannot have $\Theta(R') = R$.

**Theorem 2.** There are $(L+1)$ semi-regular subalgebras of length $L$ which cannot be pairwise connected by *-automorphisms of $\mathfrak{A}$. Specifically, these have product types $\alpha = 0, 1, 2, \cdots, L$.

We give an indication of the proof, which is constructive and depends on the results of [5]. For each $n = 1, 2, 3, \cdots$, the matrix units (of all the $2^p$ by $2^p$ matrix algebras, where $p$ is an odd multiple of $n$) are divided into $n$ orthogonal sets. These are called $\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_n$, and the set $\mathfrak{A}_g = \bigcup_{i=0}^n \mathfrak{A}_i$. The ring $R(\mathfrak{C}_g)$ is defined as the weak closure of the algebra generated by matrix units in $\mathfrak{C}_g$. Then for each $n$ and for $0 \leq \alpha \leq n - \alpha$, we construct $R_n(\alpha)$, a semi-regular subalgebra. The chain for $R_n(\alpha)$ is such that $N^t(P_n(\alpha)) = R(\mathfrak{C}_g)$ for $0 \leq t \leq \alpha$ and $N^s(P_n(\alpha)) = R(\mathfrak{C}_{\alpha+s})$ for $\alpha \leq s \leq n - \alpha$. Since $N^{n-\alpha}(P_n(\alpha)) = R(\mathfrak{C}_n) = \mathfrak{A}$, we have $L = n - \alpha$.

But these properties are sufficient to show that $R_n(\alpha)$ has product type $\alpha$. So for any $L = 1, 2, 3, \cdots$, we can take $\alpha = 0, 1, \cdots, L$, and $n = \alpha + L$. Then, by Theorem 1, there does not exist an *-automorphism $\Theta$ of $\mathfrak{A}$ such that $\Theta(R_{\alpha+L}(\alpha')) = R_{\alpha+L}(\alpha')$ when $\alpha \neq \alpha'$.

A generalization of the concept of product type permits one to construct $2^L$ nonisomorphic semi-regular maximal abelian subalgebras of every length $L$. However, the construction becomes extremely involved.
PURE SUBGROUPS HAVING PRESCRIBED SOCLES

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Let \( B = \sum B_n \) be a direct sum of cyclic groups where, for each positive integer \( n \), \( B_n = \sum C(p^n) \) is zero or homogeneous of degree \( p^n \) where \( p \) is a fixed prime. Denote by \( \bar{B} \) the torsion completion of \( B \) in the \( p \)-adic topology. Following established terminology [1], we refer to \( \bar{B} \) as the closed primary groups with basic subgroup \( B \). A primary group \( G \) is said to be pure-complete if each subsocle of \( G \) supports a pure subgroup of \( G \). A semi-complete group was defined by Kolettis in [6] to be a primary group which is the direct sum of a closed group and a direct sum of cyclic groups.

For a particular \( B \), I exhibited in [3] nonisomorphic pure subgroups \( H \) and \( K \) of \( \bar{B} \) having the same socle. Using this example, Megibben [7] was the first to show the existence of a primary group without elements of infinite height which is not pure-complete. We mention that each semi-complete group is pure-complete [4]. The purpose of this note is to announce the following theorem and corollaries; proofs will appear in another paper.

**Theorem.** Suppose that \( B \) is unbounded and countable and that \( S \) is any proper dense subsocle of \( \bar{B} \) such that \( |S| = 2^{\aleph_0} \). Then \( S \) supports more than \( 2^{\aleph_0} \) pure subgroups of \( \bar{B} \) which are isomorphically distinct.

The theorem has the following implications.

**Corollary 1.** Suppose that \( B \) is unbounded and countable and that