

DIMENSION OF METRIC SPACES AND HILBERT'S PROBLEM 13¹

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In 1957 A. N. Kolmogorov [1] and V. I. Arnol'd [2] obtained the following result (answering Hilbert's conjecture in the negative):

THEOREM. *For every integer $n \geq 2$ there exist continuous real functions ψ^{pq} , for $p=1, 2, \dots, n$ and $q=1, 2, \dots, 2n+1$, defined on the unit interval $E^1 = [0, 1]$, such that every continuous real function f , defined on the n -dimensional unit cube E^n , is representable in the form*

$$f(x_1, \dots, x_n) = \sum_{q=1}^{2n+1} \chi_q \left[\sum_{p=1}^n \psi^{pq}(x_p) \right],$$

where the functions χ_q are real and continuous.

The proof of the theorem relies on two properties of E^1 , namely, E^1 is compact and of dimension 1. (By dimension we shall always mean covering dimension.) This paper generalizes the work of Kolmogorov and Arnol'd to obtain the following result:

THEOREM 2. *For $p=1, 2, \dots, m$ let X^p be a compact metric space of finite dimension d_p , and let $n = \sum_{p=1}^m d_p$. There exist continuous functions $\psi^{pq}: X^p \rightarrow [0, 1]$, for $p=1, \dots, m$ and $q=1, 2, \dots, 2n+1$, such that every continuous real function f defined on $\prod_{p=1}^m X^p$ is representable in the form*

$$f(x_1, \dots, x_m) = \sum_{q=1}^{2n+1} \chi_q \left[\sum_{p=1}^m \psi^{pq}(x_p) \right],$$

where the functions χ_q are real and continuous.

The proof of Theorem 2 makes use of the following new characterization of dimension of metric spaces which is of interest in itself.

THEOREM 1. *A metric space X is of dimension $\leq n$ if and only if for each open cover \mathcal{C} of X and each integer $k \geq n+1$ there exist k discrete families of open sets $\mathcal{U}_1, \dots, \mathcal{U}_k$ such that the union of any $n+1$ of the \mathcal{U}_i is a cover of X which refines \mathcal{C} .*

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By a discrete family of sets we mean a family such that each point has a neighborhood which meets at most one member of the family. For dimension 0 Theorem 1 reduces to the following result, which is well known: A metric space X is of dimension ≤ 0 if and only if each open cover of X may be refined by a discrete family of open sets.

This note presents brief proofs of Theorems 1 and 2.

PROOF OF THEOREM 1. It suffices to show that if X is of dimension $\leq n$, then for each open cover \mathcal{C} of X and each integer $k \geq n+1$ there exist k discrete families of open sets $\mathfrak{U}_1, \dots, \mathfrak{U}_k$ which refine \mathcal{C} , any $n+1$ of which cover X . We prove this by induction on k . Let X be of dimension $\leq n$ and let $\mathcal{C} = \{C_\alpha; \alpha \in I\}$ be an open cover of X . By passing to a refinement if necessary, we may suppose that \mathcal{C} is locally finite. We may write X as $X = \bigcup_{i=1}^{n+1} X_i$ where each X_i is a subspace of dimension 0.

Let $\mathcal{C}_i = \{C_\alpha \cap X_i; \alpha \in I\}$. \mathcal{C}_i is an X_i -open cover of X_i . By the case of dimension 0 there exists for each i a disjoint family \mathfrak{V}_i of X_i -open sets which covers X_i and refines \mathcal{C}_i .

Let I be well ordered by $<$. For each $\alpha \in I$ and $i = 1, \dots, n+1$, let $W_\alpha^i = \bigcup \{V \in \mathfrak{V}_i; V \subset C_\alpha \cap X_i \text{ and for each } \beta < \alpha, V \not\subset C_\beta \cap X_i\}$, and let $\mathfrak{W}_i = \{W_\alpha^i; \alpha \in I\}$. Distinct members of \mathfrak{W}_i are disjoint.

Let $Z_\alpha^i = \{x \in C_\alpha; d(x, W_\alpha^i) < d(x, \bigcup_{\beta < \alpha} W_\beta^i)\}$ and let $\mathfrak{Z}_i = \{Z_\alpha^i; \alpha \in I\}$. $W_\alpha^i \subset Z_\alpha^i \subset C_\alpha$ for each i and α . Each \mathfrak{Z}_i is a locally finite family of disjoint open sets of X which refines \mathcal{C} and covers X_i . It follows that $\bigcup_{i=1}^{n+1} \mathfrak{Z}_i$ covers X .

As before, there exist closed sets $D_\alpha^i \subset Z_\alpha^i$ such that $\{D_\alpha^i; \alpha \in I; i = 1, \dots, n+1\}$ covers X . Choose open sets U_α^i such that $D_\alpha^i \subset U_\alpha^i \subset \text{Cl}(U_\alpha^i) \subset Z_\alpha^i$, and let $\mathfrak{U}_i = \{U_\alpha^i; \alpha \in I\}$. Each \mathfrak{U}_i is discrete and refines \mathcal{C} , and $\mathfrak{U}_1, \dots, \mathfrak{U}_{n+1}$ cover X .

Suppose now that $k > n+1$ and that $\mathfrak{U}_1, \dots, \mathfrak{U}_{k-1}$ are discrete families of open sets which refine \mathcal{C} , any $n+1$ of which cover X . We will construct a subset A of X and a discrete family \mathfrak{U}_k of open sets which refines \mathcal{C} , such that any n of the families $\mathfrak{U}_1, \dots, \mathfrak{U}_{k-1}$ cover $X - A$ and \mathfrak{U}_k covers A .

Let $\mathfrak{A} = \{\gamma = (\gamma_1, \dots, \gamma_n); 1 \leq \gamma_1 < \gamma_2 < \dots < \gamma_n \leq k-1\}$. For $\gamma \in \mathfrak{A}$, let $A_\gamma = \bigcap_{i=1}^n (X - \bigcup \mathfrak{U}_{\gamma_i})$, and $A = \bigcup_{\gamma \in \mathfrak{A}} A_\gamma$. Each A_γ is closed and $A_\gamma \cap A_\delta = \emptyset$ for $\gamma \neq \delta$. Hence there exist open sets B_γ such that $A_\gamma \subset B_\gamma$ and $\text{Cl}(B_\gamma) \cap \text{Cl}(B_\delta) = \emptyset$ for $\gamma \neq \delta$. For $\gamma \in \mathfrak{A}$, there exists a positive integer $j_\gamma \leq k-1$ such that $j_\gamma \notin \{\gamma_i; i = 1, 2, \dots, n\}$. \mathfrak{U}_{j_γ} covers A_γ . Let $\mathfrak{U}_k = \{U \cap B_\gamma; \gamma \in \mathfrak{A} \text{ and } U \in \mathfrak{U}_{j_\gamma}\}$. \mathfrak{U}_k is a discrete family of open sets which covers A and refines \mathcal{C} . Thus the proof of Theorem 1 is complete.

PROOF OF THEOREM 2. For each integer $p, p = 1, \dots, m$, each

integer $q, q=1, 2, \dots, 2n+1$, and each integer $k, k=1, 2, \dots$, there exist positive real numbers γ_k and ϵ_k , distinct positive prime numbers r_k^{2q} , discrete families S_k^{2q} of open sets of X^p , and continuous functions $f_k^{2q}: X^p \rightarrow [0, 1]$ such that:

- (1) $\lim_{k \rightarrow \infty} \gamma_k = \lim_{k \rightarrow \infty} \epsilon_k = 0$;
- (2) each member of S_k^{2q} is of diameter $\leq \gamma_k$ and for each fixed p and k any d_p+1 of the families S_k^{2q} cover X^p ;
- (3) $m\epsilon_k < 1/\prod_{p=1}^m r_k^{2q}$ for each $q=1, 2, \dots, 2n+1$;
- (4) f_k^{2q} is constant on each member of S_k^{2q} , the constant being an integral multiple of $1/r_k^{2q}$, and takes different values on distinct members of S_k^{2q} ;
- (5) For each $j < k$ and $x \in X^p, f_j^{2q}(x) \leq f_k^{2q}(x) \leq f_j^{2q}(x) + \epsilon_j - \epsilon_k$.

The $\gamma_k, \epsilon_k, r_k^{2q}, S_k^{2q}$, and f_k^{2q} are defined inductively on k .

Let $X = \prod_{p=1}^m X^p$. X is a metric space with metric

$$d((x_1, \dots, x_m), (y_1, \dots, y_m)) = \sum_{p=1}^m d(x_p, y_p).$$

For each q and k let $\mathfrak{I}_k^q = \{ \prod_{p=1}^m C^p : C^p \in S_k^{2q} \text{ for each } p \}$. Each \mathfrak{I}_k^q is a discrete family of open sets of X , and each member of \mathfrak{I}_k^q is of diameter $\leq m\gamma_k$. For each k any $n+1$ of the families \mathfrak{I}_k^q cover X .

Let $\psi^{2q}(x) = \lim_{k \rightarrow \infty} f_k^{2q}(x)$ for $x \in X^p$. For each k and each $x \in X^p, f_k^{2q}(x) \leq \psi^{2q}(x) \leq f_k^{2q}(x) + \epsilon_k$. Thus ψ^{2q} , being the uniform limit of the f_k^{2q} , is continuous.

Let $\phi^q(x_1, \dots, x_m) = \sum_{p=1}^m \psi^{2q}(x_p)$ for $(x_1, \dots, x_m) \in X$. Let $\mathfrak{U}_k^q = \{ \phi^q(C) : C \in \mathfrak{I}_k^q \}$. If $C = \prod_{p=1}^m C^p \in \mathfrak{I}_k^q$, then $\phi^q(C)$ is contained in the interval $[\sum_{p=1}^m f_k^{2q}(C^p), \sum_{p=1}^m f_k^{2q}(C^p) + m\epsilon_k]$. By condition (3) these closed intervals are disjoint for each fixed q and k . Hence each \mathfrak{U}_k^q is discrete.

Let f be a continuous real-valued function on X . For each integer $r \geq 0$ and $q=1, \dots, 2n+1$ there exists a positive integer k_r , and continuous functions $\chi_r^q: R \rightarrow R$ (R denotes the real line, $k_0=1$ and $\chi_0^q=0$ for each q) such that if $f_r(x) = \sum_{q=1}^{2n+1} \sum_{s=0}^r \chi_s^q(\phi^q(x))$ for $x \in X$ and if $M_r = \sup_{x \in X} |(f-f_r)(x)|$, then:

- (6) $k_{r+1} > k_r$;
- (7) if $d(a, b) < m\gamma_{k_{r+1}}$, then $|(f-f_r)(a) - (f-f_r)(b)| < (2n+2)^{-1}M_r$;
- (8) χ_{r+1}^q is constant on each member of $\mathfrak{U}_{k_{r+1}}^q$, its value on $\phi^q(C)$ for $C \in \mathfrak{I}_{k_{r+1}}^q$ being $(n+1)^{-1}(f-f_r)(y)$ for some arbitrarily chosen element y of C ;
- (9) $|\chi_{r+1}^q(a)| \leq (n+1)^{-1}M_r$ for each $a \in R$.

The k_r and χ_r^q are defined inductively on r . It is easily deduced from (7) and (8) that

(10) $|(n+1)^{-1}(f-f_r)(x) - \chi_{r+1}^q(\phi^q(x))| < (n+1)^{-1}(2n+2)^{-1}M_r$ for $x \in \bigcup \{T: T \in \mathfrak{I}_{r+1}^q\}$.

For each $x \in X$ there are at least $n+1$ distinct values of q such that that $x \in \bigcup \{T: T \in \mathfrak{I}_{r+1}^q\}$. Adding (10) for $n+1$ values of q and (9) for the other n values of q yields

$$|(f - f_{r+1})(x)| = \left| (f - f_r)(x) - \sum_{q=1}^{2n+1} \chi_{r+1}^q(\phi^q(x)) \right| < \frac{2n+1}{2n+2} M_r.$$

Then $M_{r+1} < (2n+1)(2n+2)^{-1}M_r$, so $M_r < ((2n+1)(2n+2)^{-1})^r M_0$ for each r and $\lim_{r \rightarrow \infty} M_r = 0$. Hence $f(x) = \lim_{r \rightarrow \infty} f_r(x)$ for all $x \in X$. Moreover, by condition (9) the functions $\sum_{s=0}^n \chi_s^q$ converge uniformly for each q to a continuous function $\chi^q: R \rightarrow R$ and

$$f(x) = \lim_{r \rightarrow \infty} f_r(x) = \lim_{r \rightarrow \infty} \sum_{q=1}^{2n+1} \sum_{s=0}^r \chi_s^q(\phi^q(x)) = \sum_{q=1}^{2n+1} \chi^q(\phi^q(x)).$$

This completes the proof of Theorem 2.

REFERENCES

1. A. N. Kolmogorov, *Representation of functions of many variables*, Dokl. Akad. Nauk 114 (1957), 953-956; English transl., Amer. Math. Soc. Transl. (2) 17 (1961), 369-373.
2. V. I. Arnol'd, *On functions of three variables*, Dokl. Akad. Nauk 114 (1957), 679-681; English transl., Amer. Math. Soc. Transl. (2) 28 (1963), 51-54.

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