THE EXISTENCE OF COMPLETE CYCLES IN REPEATED LINE-GRAPHS\textsuperscript{1}

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With every nonempty ordinary graph \( G \) there is associated a graph \( L(G) \), called the line-graph of \( G \), whose points are in one-to-one correspondence with the lines of \( G \) and such that two points are adjacent in \( L(G) \) if and only if the corresponding lines of \( G \) are adjacent. By \( L^2(G) \), we shall mean \( L(L(G)) \); and, in general, \( L^k(G) \) will denote \( L(L^{k-1}(G)) \) for \( k \geq 1 \), where \( L^1(G) \) and \( L^0(G) \) stand for \( L(G) \) and \( G \), respectively. The graphs \( L(G) \), \( L^2(G) \), \( L^3(G) \), etc. are referred to as the repeated line-graphs of \( G \). A complete cycle (or hamiltonian cycle) in a (connected) graph \( G \) is a cycle containing all the points of \( G \). The purpose of this note is to outline a proof of the following result, a complete proof of which will be published elsewhere.

\textbf{Theorem 1.} If \( G \) is a nontrivial connected graph of order \( p \) (having \( p \) points), and if \( G \) is not a path, then \( L^n(G) \) contains a complete cycle for all \( n \geq p - 3 \). Furthermore, the number \( p - 3 \) cannot, in general, be improved.

A graph \( G \) having \( q \) lines, where \( q \geq 3 \), is called sequential if the lines of \( G \) can be ordered as \( x_0, x_1, x_2, \ldots, x_{q-1}, x_q = x_0 \) so that \( x_i \) and \( x_{i+1}, i = 0, 1, \ldots, q - 1 \), are adjacent. The next theorem follows immediately.

\textbf{Theorem 2.} A necessary and sufficient condition that the line-graph \( L(G) \) of a graph \( G \) contain a complete cycle is that \( G \) be a sequential graph.

If a graph \( G \) contains a complete cycle \( C \), then the lines of \( C \) can be arranged in a cyclic fashion. By an appropriate “interspersing” of the lines not on \( C \) (if any) among the lines which are on \( C \), we can produce an ordering of all the lines of \( G \) as needed to show that \( G \) is sequential. This fact coupled with Theorem 2 gives the next result.

\textbf{Theorem 3.} If a graph \( G \) contains a complete cycle, then \( L(G) \) also contains a complete cycle.

\textbf{Corollary.} If a graph \( G \) contains a complete cycle, then \( L^n(G) \) contains a complete cycle for all \( n \geq 1 \).

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The following two lemmas can be quickly established.

**Lemma 1.** If a graph $G$ has a cycle $C$ with the property that every line of $G$ is incident with at least one point of $C$, then $L(G)$ contains a complete cycle.

**Lemma 2.** Let $G$ be a graph consisting of a cycle $C$ and its diagonals (a diagonal of $C$ being a line which is not on $C$ but which is incident with two points of $C$) and $m$ paths $P_1, P_2, \ldots, P_m$, where (i) each path has precisely one endpoint in common with $C$ and (ii) for $i \neq j$, $P_i$ and $P_j$ are disjoint except possibly having an endpoint in common if this point is also common to $C$. Then, if the maximum of the lengths of the $P_i$ is $M$, $L^n(G)$ contains a complete cycle for all $n \geq M$.

The proof of Theorem 1 is by induction on $p$ with the graphs having order 3, 4, or 5 treated individually. It is assumed then that for all connected graphs $G'$ which are not paths and which have order $s$, where $s < p$ and $p \geq 6$, $L^n(G')$ contains a complete cycle for each $n \geq s - 3$. The proof involves showing that if $G$ is a graph which is not a path and which has order $p$, then $L^{p-1}(G)$ is a sequential graph so that $L^{p-1}(G)$ contains a complete cycle (by Theorem 2) and $L^n(G)$ contains a complete cycle for all $n \geq p - 3$ (by the corollary to Theorem 3).

If $G$ is a cycle, the result follows directly, so without losing generality, we assume that $G$ contains a point $v$ having degree 3 or more. Let $H$ denote the connected star subgraph whose lines are all those incident with $v$, and let $Q$ denote the subgraph whose point set consists of all the points of $G$ different from $v$ and whose lines are all those which are in $G$ but not in $H$. $H$ and $Q$ have deg $v$ points in common but are line disjoint. We denote the components of $Q$ by $G_1, G_2, \ldots, G_k$.

$L(H)$ is a complete subgraph of $L(G)$ and so has a cycle containing all the points of $L(H)$. If $J_1$ denotes $L(H)$ plus all those lines in $L(G)$ incident with one point of $L(H)$, then, by Lemma 1, $H_1 = L(J_1)$ has a cycle containing all the points of $H_1$. We let $J_2$ denote $L(H_1)$ plus any lines of $L^2(G)$ incident with a point of $L(H_1)$ and let $H_2 = L(J_2)$. Once again, by Lemma 1, $H_2$ has a cycle containing all the points of $H_2$. $J_i$ and $H_i$, $i = 3, 4, \ldots$, are defined analogously, and each $H_i$ has a cycle containing all the points of $H_i$.

Two cases are considered: (1) All the $G_i$ are paths or isolated points, and (2) there is at least one $G_i$ different from a path or an isolated point. In the first case, it follows, with the aid of Lemma 2, that $L^{p-4}(G)$ contains a complete cycle so that $L^{p-2}(G)$ contains such a cycle also.
In the second case, we assume that the first \( t \) components, \( 1 \leq t \leq k \), of \( G_1, G_2, \ldots, G_k \) are not paths or isolated points. Clearly, each of the components \( G_1, G_2, \ldots, G_t \) has at least 3 points. If \( t < k \), the paths (or isolated points) \( G_{t+1}, \ldots, G_k \) have orders at most \( p-4 \), and it is easily seen that for these components, \( L^{p-4}(G_i) \) does not exist. \( L^{p-4}(G) \) can thus be expressed as the pairwise line disjoint sum of the graphs \( J_{p-4}, L^{p-4}(G_1), L^{p-4}(G_2), \ldots, L^{p-4}(G_t) \), where each of the graphs \( L^{p-4}(G_i), i = 1, 2, \ldots, t \), has a cycle containing all the points of \( L^{p-4}(G_i) \) by the inductive hypothesis.

Since \( p \geq 6 \), it can be shown that for each \( i = 1, 2, \ldots, t \), there is a point \( u_i \) in \( H_{p-4} \) adjacent to both endpoints of a line in \( L^{p-4}(G_i) \). Using this result, we produce a suitable ordering of the lines of \( L^{p-4}(G) \) thereby showing it to be a sequential graph.

Theorem 1 permits us to make the following definition.

**Definition.** Let \( G \) be a nontrivial connected graph which is different from a path. The **hamiltonian index** of \( G \), denoted by \( h(G) \), is the smallest nonnegative integer \( n \) such that \( L^n(G) \) contains a complete cycle.

It now follows immediately that a graph contains a hamiltonian cycle if and only if its hamiltonian index is zero. Theorem 1 may now be restated in the following way. If \( G \) is a nontrivial connected graph of order \( p \) which is not a path, then \( h(G) \) exists and \( h(G) \leq p - 3 \). To show that the bound given in Theorem 1 cannot be improved, we note that for every \( p \geq 3 \), there are graphs whose hamiltonian indices are \( p - 3 \). The graphs \( G_1 \) and \( G_2 \) shown in Figure 1 have hamiltonian indices equal to \( p - 3 \).

![Figure 1](image-url)