CONCERNING A CONJECTURE OF WHYBURN ON LIGHT OPEN MAPPINGS

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Introduction. Some important and fundamental theorems in complex analysis are simple consequences of theorems in the theory of light open mappings for 2-manifolds. This rather complete theory is largely the work of G. T. Whyburn [9], [10], [11]. One theorem which includes the well-known theorems of Darboux [4] and Stoïlow [8] is the following:

THEOREM (WHYBURN). *Suppose that f is a light open mapping of a disk A (topological 2-cell) onto a disk B such that* (a) $f(\text{Int } A) = \text{Int } B$ *and* (b) $f|\text{Bd } A$ *is a homeomorphism of* Bd A *onto* Bd B. *Then f is a homeomorphism.*

In his paper [12], Whyburn has conjectured that if in the above theorem each of A and B is a topological n-cell, then f is a homeomorphism. This is an extremely difficult problem. One result of this announcement provides an affirmative answer for special cases of this conjecture. Church and Hemmingsen [1], [2], [3] have made significant contributions on related problems. Meisters and Olech [7] have some results for very special types of light open mappings; namely, either locally 1-1 maps or locally 1-1 maps except on discrete sets of a certain type.

Here, each mapping is continuous and each space is metric. A mapping f of a space X into a space Y is light iff $f^{-1}f(x)$ is totally disconnected for each x in X. And, f is open iff for each U open in X, $f(U)$ is open relative to $f(X)$.

Suppose that f is a light mapping of a space X into a space Y. We shall say that the *singular set* S_f of f is the set of points x in X such that f is not locally 1-1 at x; i.e., there is no set U open in X and containing x such that $f|U$ is 1-1. We consider here only mappings f which *preserve both the boundary and the interior of* X (both of which are assumed to be nonempty).

THEOREM 1. *Suppose that X is a compact subset of a metric space M,* Bd $X \neq 0$, Int $X \neq 0$, *and f is a light open mapping of X into M such that* (1) $f(\text{Int } X) = \text{Int } f(X)$, (2) $f(\text{Bd } X) = \text{Bd } f(X)$, (3) *the singular*

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set S_f has the property that $f(S_f)$ does not contain a nonempty set open relative to $f(X)$, (4) *$f(S_f)$ does not separate $f(X)$, and* (5) *there exists a nonempty U in X open relative to X such that $f|U$ is* 1-1 *and $f^{-1}f(U) = U$. Then f is a homeomorphism.*

This theorem is a generalization of theorems due to Meisters and Olech [6]. We use some techniques of theirs and also theorems from Whyburn's theory of light open mappings.

Let P denote the set of all points y in $f(X)$ such that $f^{-1}(y)$ is nondegenerate. Now, f is a homeomorphism iff P is empty. We shall show that P is empty.

LEMMA 1. *The set P is open relative to $f(X)$ and contains a nonempty open set if P is nonempty.*

LEMMA 2. *The set $A = f(S_f) \cup P$ is closed and therefore compact.*

PROOF. Suppose that y is a limit point of A but $y \in f(X) - A$. Clearly, y is a limit point of $P - f(S_f)$. Now, $f^{-1}(y)$ is a point x and f is locally 1-1 at x. Hence, there is a *nbhd* N_x of x such that $f|N_x$ is 1-1, $f^{-1}f(N_x) = N_x$, and $f(N_x)$ is open relative to $f(X)$. This involves a contradiction.

PROOF OF THEOREM 1. Suppose that P is nonempty. It follows that $f(S_f) \cap \text{Int } f(X) = f(S_f \cap \text{Int } X)$. Now, $f(S_f) = f(S_f \cap \text{Bd } X) \cup f(S_f \cap \text{Int } X)$. Also, $f(X) - f(S_f) = \{[f(X) - f(S_f)] \cap P\} \cup \{[f(X) - f(S_f)] \cap [f(X) - P]\}$. Furthermore, $[f(X) - f(S_f)] \cap P$ is open relative to $f(X)$.

By Lemma 2, $f(S_f) \cup P$ is closed. Therefore, $B = f(X) - [f(S_f) \cup P] = [f(X) - f(S_f)] \cap [f(X) - P]$ is open in $f(X)$. Both B and $f(X) - f(S_f)$ are nonempty. Since $f(X) - f(S_f)$ is connected, $[f(X) - f(S_f)] \cap P = 0$ and, consequently, $P = 0$. We have a contradiction to our assumption that f is not 1-1 and the theorem is proved.

COROLLARY 1. *Suppose that X is a compact proper subset of an n-manifold M^n with* Int $X \neq 0$. *Furthermore, f is a local homeomorphism of X into M^n such that* (1) *$f(X)$ is connected and* (2) *there is some set U in X open relative to X such that $f^{-1}f(U) = U$ and $f|U$ is* 1-1. *Then f is a homeomorphism.*

COROLLARY 2. *Suppose that X is a compact subset of E^n with* Int $X \neq 0$, *X = closure of* Int X, Int $f(X)$ *is connected, and that f is a light open mapping of X into E^n such that* (1) *$f|$*Int X *is locally* 1-1, (2) $f(\text{Int } X) = \text{Int } f(X)$, (3) $\text{Bd } f(X) = f(\text{Bd } X)$, *and* (4) *there is U open relative to X such that $f^{-1}f(U) = U$ and $f|U$ is* 1-1. *Then f is a homeomorphism.*

Light open mappings on n-cells. Now, we are ready to give an affirmative answer to some *special cases* of Whyburn's conjecture. Consider the following theorems.

THEOREM 2. *Suppose that f is a light open mapping of an n-cell A (unit ball in E^n) onto an n-cell B (another unit ball) such that* (1) $f^{-1}f(\text{Bd } A) = \text{Bd } A$, (2) $f(\text{Bd } A) = \text{Bd } B$, (3) *dimension* $f(S_f) < n$, (4) $B - f(S_f)$ *is connected, and* (5) *there is V in B open relative to B such that $f|f^{-1}(V)$ is 1-1. Then f is a homeomorphism.*

Theorem 2 is actually a corollary of Theorem 1.

THEOREM 3. *Suppose that f is a light open mapping of an n-cell A (unit ball in E^n) onto an n-cell B such that* (1) $f^{-1}f(\text{Bd } A) = \text{Bd } A$, (2) $f(\text{Bd } A) = \text{Bd } B$, (3) $f|S_f$ *is 1-1, and* (4) *for each component C of $B - f(S_f)$, there is V in C open relative to B such that $f|f^{-1}(V)$ is 1-1. Then f is a homeomorphism.*

PROOF. Clearly f (Int A) = Int B. Since B is locally connected, there are at most a countable number of components $C_1, C_2, C_3, \cdots$, of $B - f(S_f)$. For each i, $f^{-1}(C_i)$ is connected. Denote it by K_i. Furthermore, $f(K_i) = C_i$.

Now, $f|K_i$ is locally 1-1. Also, $f(\overline{K}_i) = \overline{C}_i$ where $\overline{D}$ denotes the closure of D, and $f^{-1}(\overline{C}_i) = \overline{K}_i$. It follows that $f(\text{Bd } \overline{K}_i) = \text{Bd } f(\overline{K}_i)$, $f(\text{Int } \overline{K}_i) = \text{Int} f(\overline{K}_i)$, and $f|\overline{K}_i$ is a light open mapping of $\overline{K}_i$ onto $\overline{C}_i$.

Apply Corollary 2 where $\overline{K}_i$ replaces X. Thus, $f|\overline{K}_i$ is a homeomorphism. Let $S = \bigcup_i \overline{K}_i$. Each point of $f(S_f)$ is a limit point of $B - f(S_f)$. Thus, $B = \bigcup_i \overline{C}_i$. It follows that $f(S) = B$ and that $f|S$ is a homeomorphism of S onto B. We conclude that $S = A$ and that f is a homeomorphism.

Questions. Suppose that condition (3), namely $f|S_f$ is 1-1, is omitted from the hypothesis of Theorem 3. Is the resulting theorem true? Condition (3) may be weakened slightly as indicated in Theorem 4 below. Suppose that f is a light open mapping of an n-cell A onto an n-cell B. Does $f(S_f)$ contain an open set? This question has remained unsolved for several years (cf. [1], [2], [3]). If condition (3) that $f(S_f)$ fails to contain an open set is omitted from Theorem 1, is the resulting theorem true?

A generalization of Theorems 1 *and* 3. In Theorem 1, we require that $f(S_f)$ fail to separate $f(X)$ while in Theorem 3, we permit a separation but require that $f|S_f$ be 1-1. This may be weakened further.

THEOREM 4. *Suppose that X is a compact subset of a metric space M,* Bd $X \neq 0$, Int $X \neq 0$, *and f is a light open mapping of X into M such* *that* (1) $f(\text{Int } X) = \text{Int } f(X)$, (2) $f(\text{Bd } X) = \text{Bd } f(X)$, (3) *if* $p \in f(S_f)$ *where* S_f *is the singular set of* f, *then* p *is in the boundary of some component of* $f(X) = f(S_f)$, (4) *if* C *is a component of* $f(X) - f(S_f)$, *then* $f | f^{-1}[\bar{C} \cap f(S_f)]$ *is* 1-1, *and* (5) *for each component* K *of* $f(X) - f(S_f)$, *there is* V *in* K *open relative to* $f(X)$ *such that* $f | f^{-1}(V)$ *is* 1-1. *Then* f *is a homeomorphism.*

A proof of Theorem 4 may be obtained in a manner similar to that for Theorem 3.

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