SOME SPACES WHOSE PRODUCT WITH $E^1$ IS $E^4$

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1. Introduction. If $A$ is a collection of subsets of $E^3$, then $A^* = \bigcup \{ a \mid a \in A \}$. A sequence $A_i$, $i = 1, 2, 3, \ldots$, of locally finite disjoint collections of subsets of $E^3$ is trivial if $A^*_{i+1} \subset \text{Int}(A^*_i)$, each element of $A_i$ is a cube with handles semi-linearly imbedded in $E^3$, and the inclusion map $j: a' \to a$, where $a' \subset a \in A_i$ and $a' \in A_{i+1}$, is null homotopic.

If $A_i$, $i = 1, 2, \ldots$, is a trivial sequence let $G$ be the set of points of $E^3 - \bigcap A^*_j$ and components of $\bigcap A^*_j$. Let $X$ be the corresponding decomposition space. The main result, Theorem 2, may now be stated.

THEOREM 2. If each element of $A_i$, $i = 1, 2, \ldots$, is a solid torus, then $X \times E = E^4$.

This theorem is parallel to results in [1], [3], [4] and others. The proof is similar to that given in [4].

2. Some useful maps. Let $D = \{ z \mid z \in E^2 \text{ and } |z| \leq 1 \}, S = \{ z \mid z \in E^2 \text{ and } |z| = 1 \}, D_1 = \{ z \mid z \in E^2 \text{ and } |z| \leq 1/2 \}, T = D \times S$ and $B = D_1 \times S \subset T$. Let $p: D_1 \times E \to B$ be the universal covering of $B$ where $p$ is given by $p(x, t) = (x, e^{it})$ for $x \in D_1$, $t \in E$. Let $h: D_1 \times E \to T \times E$ by $h(x, t) = (x, e^{it}, t)$ and $q: T \times E \to T$ by $q(x, s, t) = (x, s)$ where $x \in D$, $s \in S$ and $t \in E$. Hence $qh(x, t) = p(x, t)$.

Let $B'$ be a finite subcomplex of $\text{Int}(B)$ such that the inclusion map $j: B' \to \text{Int}(B)$ is null homotopic. Using the homotopy lifting theorem, there exists $j^*: B' \to D_1 \times E$ such that:

$$
\begin{array}{ccc}
D_1 \times E & \to & T \times E \\
j^* & \uparrow & \downarrow p \\
B' & \to & B \subset T \\
j & & \end{array}
$$

is commutative and both $j^*$ and $h$ are homeomorphisms.

If $u \in B'$, $hj^*(u) = (u, \psi(u))$ where $\psi: B' \to E$. If $(x, s) \in B'$ where $x \in D_1$, $s \in S$, then $j^*(x, s) = (x, w(x, s))$ where $w: B' \to E$. By commutativity, $\psi = w$.

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Since $B'$ is compact, $\psi(B') \subset [-\alpha, \alpha]$, for some $\alpha \in \mathbb{E}$. We extend $\psi$ to a map $\Psi: T \to [-\alpha, \alpha]$ by:

\[
\Psi(u) = \psi(u) \quad \text{for } u \in B',
\]

\[
\Psi(u) = 0 \quad \text{for } u \in T - B,
\]

$\Psi(u)$ is determined by Tietze's extension theorem otherwise.

Define the homeomorphism $\lambda: T \times E \to T \times E$ by $\lambda(u, t) = (u, \Psi(u) + t)$. We call $\lambda$ the lifting map; it will be used later.

Let $T_\phi: T \to T$ be a homeomorphism of $T$ onto $T$ defined for each real number $\phi$ by the following:

\[
T_\phi(x, s) = \begin{cases} 
(x, s) & (x, s) \in \text{Bd}(T), \\
(x, se^{(-2\pi|s| - 2\phi)}) & (x, s) \in T - (B \cup \text{Bd}(T)), \\
(x, se^{-i\phi}) & (x, s) \in B
\end{cases}
\]

where $x \in D$ and $s \in S$. Let $\tau: T \times E \to T \times E$ by $\tau(u, t) = (T_\phi(u), t)$. We call $\tau$ the twisting map.

Consider $qr\lambda: T \times E \to T$. For $x \in D$, $s \in S$ and $t \in E$, $qr\lambda(x, s, t) = T_\phi(x, s, t)$ if $(x, s) \in B'$ then

\[
qr\lambda(x, s, t) = (x, se^{-\psi(x, s, t)} + t) = (x, e^{-it})
\]

by $(*).$

**Lemma 1.** The homeomorphism $f = \tau\lambda: T \times E \to T \times E$ has the following properties:

1. $f =$ id. on $\text{Bd}(T \times E)$,
2. $\text{diam}(f(B' \times w)) \leq \text{diam}(D_1 \times [-\alpha, \alpha]),$
3. $f(T \times w) \subset T \times [w - \alpha, w + \alpha]$ and
4. $f$ is uniformly continuous on $B \times E$.

**Proof.** We shall only prove (4), the other three following easily from the construction. Let $\varepsilon > 0$. For the uniformly continuous map $f = f|B \times [0, 4\pi]$, let $0 < \delta < 2\pi$ be as in the definition of uniform continuity. If $u, v \in B$, $t \leq s \in E$ and $d((u, t), (v, s)) < \delta < 2\pi$, let $k$ be an integer chosen so that $0 \leq t - 2k\pi < 2\pi$ and $0 \leq s - 2k\pi < 4\pi$. By uniform continuity, $d(\tilde{f}(u, t - 2k\pi), \tilde{f}(v, s - 2k\pi)) < \varepsilon$. But $d(f(u, t), f(v, s)) = d(\tilde{f}(u, t - 2k\pi), \tilde{f}(v, s - 2k\pi))$; hence $f$ is uniformly continuous on $B \times E$.

3. **Shrinking sets in a solid torus.** Assume now that each element of $A_i$, $i = 1, 2, \cdots$, is a solid torus.

**Lemma 2.** Let $A \in A_i$, $A_0 = A_{i+1} \cap A$ and $\varepsilon > 0$. There exists a homeo-
morphism $H$ of $E^4$ onto itself such that:

1. $H = \text{id. on the complement of } A \times E$,
2. $\text{diam}(H(A_0 \times w)) < \varepsilon$,
3. $H(A \times w) \subseteq A \times [w, w + \varepsilon]$ and
4. $H$ is uniformly continuous on $A_0 \times E$.

**Proof.** Let $g'$ be a homeomorphism of $A$ onto $T$; we assume that $g'$ is chosen so that $g'(A_0) \subseteq \text{Int}(B)$. Define $g: A \times E \to T \times E$ by $g(x, t) = (g'(x), t)$. For $g^{-1}$ and $\varepsilon > 0$ let $\delta$ be as in the definition of uniform continuity. Now make the following modifications of the function $f$ of Lemma 1. The disk $D_1$ could have been chosen to be of arbitrarily small diameter; similarly the function $\Psi$ could have been chosen so that $\alpha$ would be arbitrarily small. Then we may assume that $\text{diam}(D_1 \times [-\alpha, \alpha]) < \delta$, and furthermore that $\alpha < \varepsilon/2$. Then define a homeomorphism $f': T \times E \to T \times E$ by $f'(x, t) = (y, s + \varepsilon/2)$ where $f(x, t) = (y, s)$. Defining $H = \text{id. outside } A \times E$ and $h^{-1}f'h$ elsewhere, then $H$ is as desired.


**Theorem 1.** There exists a uniformly convergent sequence of homeomorphisms of $E^3 \times E$ onto $E^3 \times E$ whose limit $f$ satisfies:

1. $f$ is 1-1 outside $\bigcap A_i \times E$,
2. $f(g \times w)$ is a point for each $g \in G$ and
3. If $g \times w \neq g' \times w'$, then $f(g \times w) \neq f(g' \times w')$ for $g \in G$.

**Theorem 2.** If each element of $A_i$, $i = 1, 2, \ldots$, is a solid torus, then $X \times E = E^4$.

Theorem 2 of [4] is a corollary of our Theorem 2; the process of lifting and twisting is essentially the method used there. One can see that although Theorem 2 is quite general, it is a special case of the following conjecture.

**Conjecture.** If each element of $A_i$, $i = 1, 2, \ldots$, is a cube with handles, then $X \times E = E^4$.

**References**


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