RESEARCH PROBLEMS

12. Richard Bellman: Matrix theory

All matrices that appear are \( N \times N \) symmetric matrices, \( N \geq 2 \), and \( A \preceq B \) signifies that \( A - B \) is nonnegative definite. Consider the set of matrices \( X \) with the property that \( X \preceq A_1 \) and \( X \preceq A_2 \) where \( A_1 \) and \( A_2 \) are two given matrices. Let us choose an element in this set (or the element) with the property that \( g(X) \), a prescribed scalar function of \( X \), is minimized. For example, \( g(X) \) might be \( \text{tr}(X) \), \( \text{tr}(X^2) \), or the largest characteristic root of \( X \).

This procedure defines a function of \( A_1 \) and \( A_2 \) which we denote by \( m(A_1, A_2) \). Similarly, we may define \( m(A_1, A_2, A_3) \), and, generally, \( m(A_1, A_2, \ldots, A_k) \) for any \( k \geq 2 \).

Can we find a function \( g(X) \) with the property that \( m(A_1, A_2, A_3) = m(A_1, m(A_2, A_3)) \)? If so, determine all such functions, and for general \( k \geq 3 \) as well. (Received April 10, 1965.)

13. Richard Bellman: Differential approximation

Let \( y(t) \) be a given vector function belonging to \( L^2(0, T) \) and let \( x(t) \) be determined as the solution of the linear vector differential equation \( x' = Ax, x(0) = c \). Under what conditions on \( c \) and \( y \) does the expression \( \int_0^T (x-y, x-y) \, dt \) possess a minimum rather than an infimum with respect to the constant matrix \( A \)? (Received May 12, 1965.)

14. Richard Bellman: A limit theorem

It is well known that if \( u_n \geq 0 \) and \( u_{m+n} \leq u_m + u_n \) for \( m, n = 0, 1, \ldots \), then \( u_n \sim nc \) as \( n \to \infty \) for some constant \( c \). Let \( u_n(p) \) be a function of \( p \) for \( p \in S \), a given set, and \( T(p) \) be a transformation with the property that \( T(p) \in S \) whenever \( p \in S, i = 1, 2 \). Suppose that

\[
    u_{m+n}(p) \leq u_m T_1(p) + u_n(T_2(p))
\]

for all \( p \in S \) and \( m, n = 0, 1, \ldots \). Under what conditions on \( T_1(p) \) and \( T_2(p) \) and \( S \) is it true that \( u_n(p) \sim ng(p) \) as \( n \to \infty \)? When is \( g(p) \) independent of \( p \)? (Received May 12, 1965.)

15. Richard Bellman: Generalized existence and uniqueness theorems

Given a second-order linear differential equation \( u'' + p(t)u' + q(t)u = 0 \), subject to various initial and boundary conditions, there are two types of problems we can consider. The first are the classical
existence and uniqueness theorems; the second are often called "inverse problems," where the problem is that of determining the properties of the coefficients from the properties of the solution. For a version of the second type, of importance in modern physics, see, for example, B. M. Levitan and M. G. Gasymov, Determination of a differential equation by two of its spectra, Russian Math. Surveys 19, No. 2 (1964), 1–64. References to earlier work by Borg, Levinson, and others will be found there.

Let us now consider a class of problems containing both of the foregoing as special cases. A simple version is the following. Suppose that $0 \leq t \leq T$, and that $S_1, S_2, S_3, S_4$ are subsets of the interval $[0, T]$. What classes of sets $S_1, S_2, S_3, S_4$, and what types of conditions on $u$ in $S_1$, $u'$ in $S_2$, $p$ in $S_3$, and $q$ in $S_4$, determine $u$, $p$, and $q$ in $[0, T]$?

Analogous problems of greater complexity and generality can be posed for higher-order and nonlinear differential equations, for partial differential equations, and for all of the classes of functional equations of analysis. In abstract form, we consider a functional equation $u = T(u, v, a)$, where $u, v$ are functions and $a$ is a vector parameter. Partial information is given concerning $u, v$, and $a$, and it is required to deduce all of the missing information. There are analogous problems for variational and control processes. (Received May 21, 1965.)

16. L. Carlitz: A Saalschützian theorem for double series

Saalschütz proved that

$$\, \, _2F_1\left[\begin{array}{c}
-m, \, a, \, b; \\
c, \, d
\end{array}\right] = \frac{(c-a)m(c-b)m}{(c-m)(c-a-b)m},$$

provided

$$c + d = a + b - m + 1.$$ 

The writer (J. London Math. Soc. 38 (1963), 415–418) proved that the series

$$S = \sum_{r=0}^{m} \sum_{s=0}^{n} \frac{(-m)_r(-n)_s(a)_{r+s}(b)_{r}(b')_s}{r!s!(c)_{r+s}(d)_r(d')_s}$$

satisfies

$$S = \frac{(b + b' - a)_{m+n}(b')_m(b)_n}{(b + b' - a)_{m+n}(b' - a)_{m}(b - a)_n}$$

provided
\[ c + d = a + b - m + 1, \]
\[ c + d' = a + b' - n + 1, \]
\[ c = b + b'. \]

Now these conditions imply

\[ (*) \quad 2c + d + d' = 2a + b + b' - m - n + 2. \]

The question arises whether \( S \) can be summed when only the condition \((*)\) is assumed. (Received May 15, 1965.)

17. Olga Taussky: *Matrix theory*

A. It is known that the \( n \times n \) hermitian matrices are closed under the Jordan product \( AB + BA \). This composition is commutative. A noncommutative composition generalizing the Jordan product can be given for general complex matrices (or matrices over an abstract field with an involution) in the following way

\[ AB + B^*A^*. \]

By \( X^* \) is meant the complex conjugate and transposed matrix. Study the structure of this generalized Jordan algebra. The idea to study this composition comes from Lyapunov's theorem which states that a matrix whose characteristic roots have positive real parts has a unique "generalized Jordan product" hermitian inverse which is a positive definite hermitian matrix (see O. Taussky, *A remark on a theorem of Lyapunov*, J. Math. Anal. Appl. 2 (1961), 105-107).

B. Let \( A \) be an \( n \times n \) matrix with complex elements and characteristic roots with negative real parts. It is known (theorem of Lyapunov) that such matrices are characterized by the fact that a positive definite matrix \( G \) exists with

\[ AG + GA^* \text{ negative definite.} \]

What is the range of \( AG + GA^* \) if \( G \) runs through all positive definite \( n \times n \) matrices? \( A^* \) is the complex conjugate and transposed matrix.

C. What can one say about pairs of matrices which can be transformed to Jordan normal form simultaneously by a similarity? (Received May 20, 1965.)