The following problem was proposed by J. E. Littlewood about 15 years ago: Let \( S(x) = \sum_{n=-\infty}^{\infty} c_n e^{inz} \) be a trigonometric series having the property that all its partial sums are positive. Is such a series necessarily a Fourier series? The purpose of this note is to show that such is not the case. It is well known that such a series must be a Fourier-Stieltjes series, and, as was shown by H. Helson, even the weaker condition

\[
(1) \quad \int |S_n(x)| \, dx < \text{const.}, \quad \left( S_n(x) = \sum_{-n}^{n} c_n e^{inz} \right)
\]

implies \( c_n = o(1) \) (cf. Zygmund [2, p. 286]). It has been shown by Mary Weiss [1] that condition (1) does not imply that \( S(x) \) is a Fourier series.

**Lemma 1.** There exists a constant \( \alpha > 0 \) such that for every \( \epsilon > 0 \) there exists a real valued trigonometric polynomial \( P_{\epsilon}(x) \), with vanishing constant coefficient, having the properties:

(i) \( |\hat{P}(j)| < \epsilon \),

(ii) \( P_{\epsilon}(x) > \alpha \) on a set of measure \( > \alpha \),

(iii) The absolute values of the partial sums of \( P_{\epsilon}(x) \) are less than \( 1/2 \).

**Proof.** There exists a constant \( C \) such that \( |(1/\sqrt{N}) \sum_{n=1}^{N} e^{in \log n} e^{inz}| < C \) (cf. Zygmund [2, p. 199]). Take \( N > \epsilon^{-2} \) and \( P_{\epsilon}(x) = \text{Re}((1/2\sqrt{N}) \sum_{n=1}^{N} e^{in \log n} e^{inz}) \). Properties (i) and (iii) are obvious. Property (ii) follows from the fact that

\[
\|P_{\epsilon}\|_{L^2} = \frac{1}{2\sqrt{2C}}, \quad \sup |P_{\epsilon}(x)| \leq \frac{1}{2}.
\]

We shall also need the following lemma:

**Lemma 2.** Let \( f_j(x) \) be real valued trigonometric polynomials satisfying:

(a) \( f_j(0) = 0 \),

(b) \( f_j(x) > \epsilon \) on a set of measure \( > \alpha \),

(c) \( |f_j(x)| < 1/2 \).

Then, if \( \lambda_j \to \infty \) fast enough, the product

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\[ \prod_{1}^{\infty} (1 - f_{j}(\lambda_{j}x)) \]

converges weakly to a singular measure.

**Proof.** Our first condition on the growth of \( \lambda_{n} \) is:

\[ \lambda_{n} > 3 \text{ times the degree of } \prod_{1}^{n-1} (1 - f_{j}(\lambda_{j}x)) \]

which implies that the constant term of \( \prod_{1}^{n} (1 - f_{j}(\lambda_{j}x)) \) is 1 for all \( n \). Since the partial products are positive, this implies that the (formal) product (2) is a Fourier-Stieltjes series of a positive measure \( \mu \). All that we have to do now is follow the lines of the proof of Theorem V.7.6, p. 209 in Zygmund [2]: We notice first that the partial products \( \prod_{1}^{n} (1 - f_{j}(\lambda_{j}x)) \) are partial sums of \( S(d\mu) \) followed by long gaps. As is well known, this implies \( \prod_{1}^{n} (1 - f_{j}(\lambda_{j}x)) \rightarrow \phi(x) \) a.e. where \( \phi(x)dx \) is the absolutely continuous part of \( \mu \); but if \( \lambda_{n} \) grows fast enough (b) implies that the only limit \( \prod_{1}^{n} (1 - f_{j}(\lambda_{j}x)) \) can converge to a.e. is zero.

**The Example.** We take

\[ S(x) = \prod_{1}^{n} (1 - P_{e_{j}}(\lambda_{j}x)). \]

The \( P_{e_{j}} \) are the polynomials defined in Lemma 1, with

\[ 0 < e_{j} < 2^{-j-2} \left\| \prod_{1}^{j-1} (1 - P_{e_{k}}(\lambda_{k}x)) \right\|^{-1} \]

(where \( \|g\|_{A} = \sum|g(n)| \) and \( \lambda_{j} \rightarrow \infty \) rapidly enough so that

(a) \( \lambda_{j} > 3 \text{ times the degree of } \prod_{1}^{j-1} (1 - P_{e_{k}}(\lambda_{k}x)) \) and

(b) \( S(x) \) is the Fourier-Stieltjes series of a singular measure

(Lemma 2).

From (a) above it follows that a partial sum of \( S(x) \) has necessarily the form \( \prod_{1}^{q} (1 - P_{e_{j}}(\lambda_{j}x)) \) times a partial sum of \( (1 - P_{e_{q+1}}(\lambda_{q+1}x)) \) plus two groups of terms each having the form

\[ P_{e_{q+1}}(k)e^{ikx} \text{ times some terms from } \prod_{1}^{q} (1 - P_{e_{j}}(\lambda_{j}x)). \]

By (iii) \( \prod_{1}^{q} (1 - P_{e_{j}}(\lambda_{j}x)) > 2^{-q} \) and the partial sums of \( (1 - P_{e_{q+1}}(\lambda_{q+1}x)) \) are >1/2 and by (4) the sum of the remaining terms is bounded by \( 2^{-q-2} \), hence the partial sums of \( S(x) \) are positive.

**References**