MODELS OF SPACE-TIME

BY W. F. EBERLEIN

Communicated by M. Kac, May 17, 1965

1. Introduction. In [1] we exhibited electron spin as a non-relativistic geometric property of (a model of) Euclidean 3-space. We now extend our model to one of space-time. The connections between 2 and 4 component spinors become lucid, while the Dirac equation and its relativistic "invariance" properties undergo a fundamental simplification and clarification.

2. Abstract space-time. We need first an axiomatic foundation strong enough to support both our mathematical considerations and their applications to physics.

DEFINITION. An n+1 dimensional space-time \((n \geq 1)\) consists of

(A) An \(n+1\) dimensional vector space \(V\) over the real numbers plus a symmetric bilinear real form \(A \cdot B\) (inner product) such that:

1. There exists a vector \(A\) with \(A \cdot A < 0\).

2. Any 2-dimensional subspace of \(V\) contains a vector \(A\) with \(A \cdot A > 0\).

(B) A set \(\chi\) of objects \(p, q, \ldots\) (points or "events") plus a mapping \((p, q) \rightarrow p - q\) of \(\chi \times \chi\) into \(V\) such that:

1. \((p - g) + (q - r) = p - r\).

2. \(p - q = 0\) implies \(p = q\).

3. Given any point \(q\) and any vector \(A\) there exists a point \(p\) with \(p - q = A\).

Any \(V\) satisfying (A) yields a model of space-time (vector space-time) on setting \(\chi = V\). The Minkowski model \(V = \chi = \mathbb{R}_{M}^{n+1}\) consists of all \(n+1\)-tuples of real numbers \(x = (x_1, \ldots, x_n, x_{n+1})\) with \(x \cdot y = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1}\). (When \(n = 3\), \(x_4 = ct\), where \(t\) is time and \(c\) is the velocity of light.) Every \(n+1\) dimensional vector space-time is isomorphic to \(\mathbb{R}_{M}^{n+1}\), but this result is physically misleading. Eventually we set \(n = 3\), \(\chi = \) the physical space-time continuum, and \(V = \mathcal{C}_4\), the spin model of (vector) space-time we shall construct.

3. The models \(\mathcal{C}_3\) and \(\mathcal{W}_4\). In [1] we defined the spin model \(\mathcal{C}_3\) of Euclidean 3-space as the vector space of self-adjoint linear transformations of trace 0 in a 2-dimensional unitary space \(H_2\) (spin space) plus the operations \(A \cdot B = (1/2)(AB + BA)\) and \(A \times B = (1/2i)\).

---

1 Work supported by National Science Foundation Grant NSF G-17774.
\( (AB - BA) \). (We identify a scalar \( c \) with \( cI \), where \( I \) is the identity transformation in \( H_2 \).) In general we denote the algebra of linear transformations in a vector space \( E \) by \( B(E) \). We summarize some results of [1] that we need:

Relative to an arbitrary orthonormal basis \( \phi_1, \phi_2 \) for \( H_2 \) any vector \( A \) in \( \mathbb{C}_3 \) has the matrix representation

\[
A \mapsto \begin{pmatrix} x_3 & x_1 - i x_2 \\ x_1 + i x_2 & -x_3 \end{pmatrix} = x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3,
\]

where

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

are the Pauli matrices. Then \( \mathbb{C}_3 \) is 3-dimensional and

\[
A \cdot A = A^2 = x_1^2 + x_2^2 + x_3^2 = - \det A.
\]

Let \( SU(2) \) denote the group of unitary transformations in \( H_2 \) of determinant 1 and \( SO(3) \), the group of rotations or orthogonal transformations of determinant 1 in \( \mathbb{C}_3 \). Given \( U \) in \( SU(2) \) set \( R_U \cdot A = U A U^{-1} \) \((A \in \mathbb{C}_3)\). Then \( R_U \) is a linear transformation in \( \mathbb{C}_3 \), and the mapping \( U \mapsto R_U \) is a 2-to-1 homomorphism of \( SU(2) \) onto \( SO(3) \).

The obvious extension of \( \mathbb{C}_3 \) is the vector space \( W_4 \) consisting of all self-adjoint linear transformations in \( H_2 \). Then for any \( A \) in \( W_4 \)

\[
A \mapsto \begin{pmatrix} x_4 & x_3 & x_1 - i x_2 \\ x_3 & x_4 & x_2 + i x_1 \\ x_1 + i x_2 & x_2 - i x_1 & -x_4 \end{pmatrix} = x_3 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 + x_4
\]

and \(-\det A = x_1^2 + x_2^2 + x_3^2 - x_4^2 \equiv A \cdot A \). \( W_4 \) is then a 3+1-dimensional vector space-time, but the corresponding inner product is hybrid:

\[
A \cdot B = \frac{1}{2} (AB + BA) - \frac{1}{2} (\text{trace } B) A - \frac{1}{2} (\text{trace } A) B.
\]

One can now extend the covering map above by setting \( SL(2, \mathbb{C}) \) = the group of linear transformations in \( H_2 \) of determinant 1, \( \mathbb{C}^\perp \) = the homogeneous proper orthochronous Lorentz group; i.e., the linear transformations in \( W_4 \) that preserve the inner product, have determinant 1, and don’t exchange past and future. Given \( S \) in \( SL(2, \mathbb{C}) \) set \( M_S A = SAS^* \) \((A \in W_4)\). Then \( M_S \) is a linear transformation in \( W_4 \), and one has the extended

**Theorem 3.1.** The mapping \( S \mapsto M_S \) is a 2-to-1 homomorphism of \( SL(2, \mathbb{C}) \) onto \( \mathbb{C}^\perp \).

This result is essentially known in matrix disguise, but the co-
ordinate-free methods of [1] afford a simpler and more incisive proof than is to be found in the literature.

Although its inner product lacks the Jordan form substituting in $C_3$, the model $W_4$ is appropriate to analysis of the Maxwell equations and the Weyl neutrino, as we shall show in a later paper.

4. The antiquaternion unit $J$. What one wants is an element $J$ in $B(H_2)$ with real square and anticommuting with $C_3$. But the only element of $B(H_2)$ that anticommutes with $C_3$ is 0. For the same reason no nonsingular $U$ in $B(H_2)$ yields the space inversion $P: RU = UA U^{-1} = -A$ ($A \in C_3$). We are thus led to the following

**Problem.** Find all antilinear transformations $J$ in $H_2$ anticommuting with $C_3$, in particular those such that $J^2 = \pm 1$.

In an equivalent guise (commutativity of $J$ with the quaternion algebra $Q = \{kU : k \geq 0, U \in SU(2)\}$ (cf. [2])) we obtained in [4] the following

**Solution.** Given an arbitrary orthonormal basis $\phi_1, \phi_2$ in $H_2$, identify a vector $x_1\phi_1 + x_2\phi_2$ with the column vector

$$
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
$$

Then every such $J$ is of the form

$$
(1) \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \omega \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},
$$

whence $J^2 = -|\omega|^2 \neq 1$ and $J^2 = -1$ iff $|\omega| = 1$—i.e., iff $J$ is anti-unitary.

The normalized $J$, $J^2 = -1$, thus obtained is unique up to a phase factor and may be identified with Wigner’s nonrelativistic time-inversion operator for particles of spin $\frac{1}{2}$, but the idea goes back to Möbius: The space inversion operator $R J A = JA J^{-1} = -A$ ($A \in C_3$) arising is independent of the scalar $\omega \neq 0$, whence one can regard (1) as an anti-projective transformation in homogeneous coordinates. Set $z = x_1/x_2$, $z' = x_1'/x_2'$ to obtain

$$
(2) \quad z' = -\bar{z}^{-1}.
$$

Now map onto the Riemann sphere, $z \rightarrow \xi$, and note that $\xi'$ is antipodal to $\xi$.

We can now rewrite the defining properties of $C_3$ as follows:

**$C_3$ consists of all $T$ in $B(H_2)$ such that**

$$
(3) \quad iT = T^* i, \quad JT = -T^* J,
$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
while the identity $A^{*}JA = (\det A)^{-1}J$ for $A$ in $B(H_2)$ translates the defining properties of $SU(2)$ into:

SU(2) consists of all $T$ in $B(H_2)$ such that

$$T^{*}iT = i,$$

$$T^{*}JT = J.$$  \(4\)

These formulae are independent of the phase factor for the normalized $J$. We now pick a distinguished $J$. This amounts to putting a complex orientation on $H_2$ (cf. [4]).

5. The spin model $\mathbb{C}_4$ and the group $\mathbb{G}_4$. Now let $E_4$ be $H_2$ considered as a real vector space plus the new inner product

$$\langle x \mid y \rangle_+ = \alpha(\langle x \mid y \rangle).$$  \(5\)

$E_4$ is a 4-dimensional Euclidean vector space. Linear and antilinear transformations in $H_2$ are then on the same footing as linear transformations in $\mathbb{C}_4$, betraying their origin only in commutativity or anticommutativity with the now distinguished linear transformation $i$. $S = T^{*}$ in $B(H_2)$ implies $S = T^{*}$ in $B(E_4)$, while the new and old trace and determinant of a $T$ from $B(H_2)$ are connected as follows:

$$\text{trace}_4 T = 2\alpha(\text{trace}_2 T),$$

$$\text{det}_4 T = |\text{det}_2 T|^2.$$  \(6\)

DEFINITION. $\mathbb{C}_4$ consists of all linear transformations in $E_4$ satisfying (3).

Clearly $\mathbb{C}_4$ is a subspace of $B(E_4)$ containing $\mathbb{C}_3$ and closed under $*$. 

THEOREM 5.1. $\mathbb{C}_4$ consists of all elements of $B(E_4)$ of the form

$$T = A + aJ \quad (A \subseteq \mathbb{C}_3, \text{a real}).$$

Then $T^2 = A^2 - a^2$ and we can set $T_1 \cdot T_2 = \frac{1}{2}(T_1 T_2 + T_2 T_1)$ to obtain a 3+1 dimensional model of vector space-time.

Let $K = (1 + J)/2^{1/2}$. Then $K$ is orthogonal, $K^2 = J$, and $K^8 = 1$.

THEOREM 5.2. The mapping $\tau: A \rightarrow KAK$ is an isomorphism of $W_4$ onto $\mathbb{C}_4$, leaving $\mathbb{C}_3$ pointwise fixed and preserving the inner product.

Since every $T$ in $B(E_4)$ admits a unique decomposition $T = T_1 + T_2$, where $T_1$, $T_2$ are respectively linear and antilinear transformations in $H_2$, the space-time $\mathbb{C}_4$ splits naturally into space and time.

DEFINITION. $\mathbb{G}_4$ consists of all linear transformations $T$ in $E_4$ satisfying (4).

THEOREM 5.3. $\mathbb{G}_4$ is a group containing $SU(2)$ and closed under $*$. 
If \( T \in \mathfrak{g}_4^+ \) set \( L_T A = T A T^{-1} \) \((A \in \mathfrak{e}_4)\). Then \( L_T \) is a linear transformation in \( \mathfrak{e}_4 \), and

**Theorem 5.4.** The mapping \( T \rightarrow L_T \) is a 2-to-1 homomorphism of \( \mathfrak{g}_4^+ \) onto \( \mathcal{L}_4^+ \).

Space-inversion \( P \) and time-reversal \( T \) arise as follows: \( P: A \rightarrow J A J^{-1}, T: A \rightarrow i A i^{-1} \). Let \( \mathfrak{g} \) be the group of linear transformations in \( E_4 \) generated by \( \mathfrak{g}_4^+ \), \( J \), and \( i \).

The connection between 2- and 4-component spinors is then contained in

**Theorem 5.5.** The mapping \( \nu: S \rightarrow K S K^{-1} \) is an isomorphism of \( \text{SL}(2, \mathbb{C}) \) onto \( \mathfrak{g}_4^+ \) leaving \( \text{SU}(2) \) pointwise fixed.

**Theorem 5.6.** The following diagram is commutative:

\[
\begin{array}{ccc}
\mathfrak{g}_4 & \xrightarrow{M_S} & \mathfrak{e}_4 \\
\downarrow{\tau} & & \downarrow{\tau} \\
\mathfrak{e}_4 & \xrightarrow{L_{K S K^{-1}}} & \mathfrak{g}_4
\end{array}
\]

Note that \( \det_4 K S K^{-1} = \det_4 S = |\det_2 S|^2 = 1 \), while \( \det J = \det K = 1 \) and \( \det_4 i = 1 \), whence \( \mathfrak{g}_4^+ \) (or \( \mathfrak{g} \)) and \( \text{SL}(2, \mathbb{C}) \) are subgroups of \( \text{SL}(4, \mathbb{R}) \) whose intersection is \( \text{SU}(2) \).

\( \mathfrak{e}_4 \) is also remarkable in that it admits an explicit coordinate-free oriented volume function \( \theta(A_1, A_2, A_3, A_4) = \frac{1}{2} \text{trace}_4 (iA_1 A_2 A_3 A_4 J) \), reducing to \((1/2i) \text{trace}_2 (A_1 A_2 A_3) = (A_1 \times A_2) \cdot A_3\) when \( A_4 = J \) and \( A_1, A_2, A_3 \) lie in \( \mathfrak{e}_3 \) (cf. [3]). Finally, the (Clifford) algebra generated by \( \mathfrak{e}_4 \) is just \( B(E_4) \).

6. **The Dirac operator.** Let \( (g_{ij}) = \text{diag}(1, 1, 1, -1) \). Then an ordered orthonormal basis \( (e) \) for \( \mathfrak{e}_4 \) is characterized by the identity

\[
(7) \quad e_i e_j + e_j e_i = 2g_{ij}.
\]

Let \( E_4^+ \) and \( \mathfrak{e}_4^+ \) be the respective complexifications of \( E_4 \) and \( \mathfrak{e}_4 \) and consider the expression \( \langle Au, v \rangle \), where \( A \) runs over \( \mathfrak{e}_4 \) and \( u, v \) run over \( E_4 \). Since this expression is real linear in \( A \), complex linear in \( u \), and complex antilinear in \( v \), there exists a unique mapping \( F: E_4^+ \times \mathcal{E}_4 \rightarrow \mathfrak{e}_4^+ \) such that

\[
(8) \quad \langle Au, v \rangle = A \cdot F(u, v),
\]
and $F(u, v)$ is complex linear in $u$ and complex antilinear in $v$. In particular, $F(u, Ju)$ lies in $\mathbb{C}_4$.

Given now any ordered o.n. basis $e_1, \cdots, e_4$ for $\mathbb{C}_4$ consider smooth functions $\psi: \mathbb{C}_4 \rightarrow \mathbb{C}_4$ and let

$$
(9) \quad (\partial_x \psi)(x) = \lim_{h \to 0} \frac{\psi(x + he) - \psi(x)}{h}.
$$

**Definition.** The Dirac operator $\mathcal{D} = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3 - e_4 \partial_4$. Then $\mathcal{D}^2 = \partial_1^2 + \partial_2^2 + \partial_3^2 - \partial_4^2$, the d'Alembertian, while the Dirac equation takes the form

$$
(10) \quad \mathcal{D} \psi + \kappa \psi = 0 \quad (\kappa = mc/\hbar),
$$

and the associated charge-current vector $-F(\psi, J\psi)$ satisfies the continuity equation

$$
(11) \quad \text{div} F(\psi, J\psi) = 0.
$$

Finally the relativistic “invariance” properties of the Dirac equation reduce to simple properties of the Dirac operator $\mathcal{D}$.

**Theorem 6.1 (Passive invariance).** $\langle \mathcal{D} \psi | u \rangle = \text{div} F(\psi, u) (u \in \mathbb{C}_4)$.

If $T \in \mathfrak{g}$, let $(T\psi)(x) = T\psi(L_{-1}x) = T\psi(T^{-1}xT)$.

**Theorem 6.2 (Active invariance).** $\mathcal{D}T = T\mathcal{D}$.

Proofs of the above theorems and some related results will appear elsewhere.

**References**


University of Rochester