MODELS OF SPACE-TIME

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Communicated by M. Kac, May 17, 1965

1. Introduction. In [1] we exhibited electron spin as a nonrelativistic geometric property of (a model of) Euclidean 3-space. We now extend our model to one of space-time. The connections between 2 and 4 component spinors become lucid, while the Dirac equation and its relativistic “invariance” properties undergo a fundamental simplification and clarification.

2. Abstract space-time. We need first an axiomatic foundation strong enough to support both our mathematical considerations and their applications to physics.

Definition. An n+1 dimensional space-time consists of
(A) An n+1 dimensional vector space V over the real numbers plus a symmetric bilinear real form A·B (inner product) such that:
1. There exists a vector A with A·A < 0.
2. Any 2-dimensional subspace of V contains a vector A with A·A > 0.

(B) A set χ of objects p, q, • • • (points or “events”) plus a mapping (p, q) → p−q of χ×χ into V such that:
1. (p−q) + (q−r) = p−r.
2. p−q = 0 implies p = q.
3. Given any point q and any vector A there exists a point p with p−q = A.

Any V satisfying (A) yields a model of space-time (vector space-time) on setting χ = V. The Minkowski model V = χ = ℝ^n+1 consists of all n+1-tuples of real numbers x = (x₁, • • • , xₙ, xₙ₊₁) with x·y = x₁y₁ + • • • + xₙyₙ − xₙ₊₁yₙ₊₁. (When n = 3, x₄ = ct, where t is time and c is the velocity of light.) Every n+1 dimensional vector space-time is isomorphic to ℝ^n+1, but this result is physically misleading. Eventually we set n = 3, χ = the physical space-time continuum, and V = ℂ₄, the spin model of (vector) space-time we shall construct.

3. The models ℂ₃ and ℂ₄. In [1] we defined the spin model ℂ₃ of Euclidean 3-space as the vector space of self-adjoint linear transformations of trace 0 in a 2-dimensional unitary space H₂ (spin space) plus the operations A·B = (1/2)(AB + BA) and A×B = (1/2i)
\( \cdot (AB - BA). \) (We identify a scalar \( c \) with \( cI \), where \( I \) is the identity transformation in \( H_2 \).) In general we denote the algebra of linear transformations in a vector space \( E \) by \( B(E) \). We summarize some results of \([1]\) that we need:

Relative to an arbitrary orthonormal basis \( \phi_1, \phi_2 \) for \( H_2 \) any vector \( A \) in \( \mathbb{C}_3 \) has the matrix representation

\[
A \mapsto \begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix} = x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3,
\]

where

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

are the Pauli matrices. Then \( \mathbb{C}_3 \) is 3-dimensional and

\[
A \cdot A = A^2 = x_1^2 + x_2^2 + x_3^2 = -\det A.
\]

Let \( SU(2) \) denote the group of unitary transformations in \( H_2 \) of determinant 1 and \( SO(3) \), the group of rotations or orthogonal transformations of determinant 1 in \( \mathbb{C}_3 \). Given \( U \) in \( SU(2) \) set \( R_U A = U A U^{-1} \) \((A \in \mathbb{C}_3)\). Then \( R_U \) is a linear transformation in \( \mathbb{C}_3 \), and the mapping \( U \mapsto R_U \) is a 2-to-1 homomorphism of \( SU(2) \) onto \( SO(3) \).

The obvious extension of \( \mathbb{C}_3 \) is the vector space \( W_4 \) consisting of all self-adjoint linear transformations in \( H_2 \). Then for any \( A \) in \( W_4 \)

\[
A \mapsto \begin{pmatrix} x_4 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_4 - x_3 \end{pmatrix} = x_1 \sigma_1 + x_2 \sigma_2 + x_3 \sigma_3 + x_4
\]

and \(-\det A = x_1^2 + x_2^2 + x_3^2 - x_4^2 \equiv A \cdot A. \) \( W_4 \) is then a 3+1-dimensional vector space-time, but the corresponding inner product is hybrid:

\[
A \cdot B = \frac{1}{2}(AB + BA) - \frac{1}{2}(\text{trace } B)A - \frac{1}{2}(\text{trace } A)B.
\]

One can now extend the covering map above by setting \( SL(2, \mathbb{C}) \) = the group of linear transformations in \( H_2 \) of determinant 1, \( \mathbb{L}_\uparrow \) = the homogeneous proper orthochronous Lorentz group; i.e., the linear transformations in \( W_4 \) that preserve the inner product, have determinant 1, and don’t exchange past and future. Given \( S \) in \( SL(2, \mathbb{C}) \) set \( M_S A = SAS^* \) \((A \in W_4)\). Then \( M_S \) is a linear transformation in \( W_4 \), and one has the extended

**Theorem 3.1.** The mapping \( S \mapsto M_S \) is a 2-to-1 homomorphism of \( SL(2, \mathbb{C}) \) onto \( \mathbb{L}_\uparrow \).

This result is essentially known in matrix disguise, but the co-
ordinate-free methods of [1] afford a simpler and more incisive proof than is to be found in the literature.

Although its inner product lacks the Jordan form substituting in \( \mathbb{C}_8 \), the model \( W_4 \) is appropriate to analysis of the Maxwell equations and the Weyl neutrino, as we shall show in a later paper.

4. The antiquaternion unit \( J \). What one wants is an element \( J \) in \( B(H_2) \) with real square and anticommuting with \( \mathbb{C}_8 \). But the only element of \( B(H_2) \) that anticommutes with \( \mathbb{C}_8 \) is 0. For the same reason no nonsingular \( U \) in \( B(H_2) \) yields the space inversion \( P : R_U A = U A U^{-1} = -A \) \( (A \in \mathbb{C}_8) \). We are thus led to the following PROBLEM. Find all antilinear transformations \( J \) in \( H_2 \) anticommuting with \( \mathbb{C}_8 \), in particular those such that \( J^2 = \pm 1 \).

In an equivalent guise (commutativity of \( J \) with the quaternion algebra \( \mathbb{Q} = \{ k U : k \geq 0, U \in \text{SU}(2) \} \) (cf. [2])) we obtained in [4] the following SOLUTION. Given an arbitrary orthonormal basis \( \phi_1, \phi_2 \) in \( H_2 \), identify a vector \( x_1 \phi_1 + x_2 \phi_2 \) with the column vector

\[
\begin{pmatrix}
x_1 \\
x_2
\end{pmatrix}
\]

Then every such \( J \) is of the form

\[
\begin{pmatrix}
x'_1 \\
x'_2
\end{pmatrix} = J \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \omega \begin{pmatrix}
x_1 \\
x_2
\end{pmatrix} = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\
x_2
\end{pmatrix},
\]

whence \( J^2 = -|\omega|^2 \neq 1 \) and \( J^2 = -1 \) iff \( |\omega| = 1 \)—i.e., iff \( J \) is anti-unitary.

The normalized \( J, J^2 = -1 \), thus obtained is unique up to a phase factor and may be identified with Wigner’s nonrelativistic time-inversion operator for particles of spin \( \frac{1}{2} \), but the idea goes back to Möbius: The space inversion operator \( R_J A = JAJ^{-1} = -A \) \( (A \in \mathbb{C}_8) \) arising is independent of the scalar \( \omega \neq 0 \), whence one can regard (1) as an anti-projective transformation in homogeneous coordinates. Set \( z = x_1/x_2 \), \( z' = x'_1/x'_2 \) to obtain

\[
z' = -\bar{z}^{-1}.
\]

Now map onto the Riemann sphere, \( z \to \xi \), and note that \( \xi' \) is antipodal to \( \xi \).

We can now rewrite the defining properties of \( \mathbb{C}_8 \) as follows:\n
\( \mathbb{C}_8 \) consists of all \( T \) in \( B(H_2) \) such that

\[
i T = T^* i, \quad J T = -T^* J,
\]

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while the identity \( A^* J A = (\det A)^{-1} J \) for \( A \) in \( B(H_2) \) translates the defining properties of \( SU(2) \) into:

**SU(2) consists of all \( T \) in \( B(H_2) \) such that**

\[
T^* i T = i, \\
T^* J T = J.
\]

These formulae are independent of the phase factor for the normalized \( J \). We now pick a distinguished \( J \). This amounts to putting a complex orientation on \( H_2 \) (cf. [4]).

5. **The spin model \( \mathfrak{e}_4 \) and the group \( G_4^+ \).** Now let \( E_4 \) be \( H_2 \) considered as a real vector space plus the new inner product

\[
\langle x | y \rangle_+ = \Re(\langle x | y \rangle).
\]

\( E_4 \) is a 4-dimensional Euclidean vector space. Linear and antilinear transformations in \( H_2 \) are then on the same footing as linear transformations in \( \mathfrak{e}_4 \), betraying their origin only in commutativity or anticommutativity with the now distinguished linear transformation \( i \). \( S = T^* \) in \( B(H_2) \) implies \( S = T^* \) in \( B(E_4) \), while the new and old trace and determinant of a \( T \) from \( B(H_2) \) are connected as follows:

\[
\text{trace}_4 T = 2 \Re(\text{trace}_2 T), \\
\text{det}_4 T = |\text{det}_2 T|^2.
\]

**Definition.** \( \mathfrak{e}_4 \) consists of all linear transformations in \( E_4 \) satisfying (3).

Clearly \( \mathfrak{e}_4 \) is a subspace of \( B(E_4) \) containing \( \mathfrak{e}_3 \) and closed under *. 

**Theorem 5.1.** \( \mathfrak{e}_4 \) consists of all elements of \( B(E_4) \) of the form

\[
T = A + aJ \ (A \subset \mathfrak{e}_3, \ a \text{ real}).
\]

Then \( T^2 = A^2 - a^2 \) and we can set \( T_1 \cdot T_2 = \frac{1}{2}(T_1 T_2 + T_2 T_1) \) to obtain a 3+1 dimensional model of vector space-time.

Let \( K = (1 + J)/2^{1/2} \). Then \( K \) is orthogonal, \( K^2 = J \), and \( K^8 = 1 \).

**Theorem 5.2.** The mapping \( \tau : A \rightarrow KAK \) is an isomorphism of \( \mathfrak{w}_4 \) onto \( \mathfrak{e}_4 \) leaving \( \mathfrak{e}_3 \) pointwise fixed and preserving the inner product.

Since every \( T \) in \( B(E_4) \) admits a unique decomposition \( T = T_1 + T_2 \), where \( T_1, T_2 \) are respectively linear and antilinear transformations in \( H_2 \), the space-time \( \mathfrak{e}_4 \) splits naturally into space and time.

**Definition.** \( G_4^+ \) consists of all linear transformations \( T \) in \( E_4 \) satisfying (4).

**Theorem 5.3.** \( G_4^+ \) is a group containing \( SU(2) \) and closed under *.
If \( T \in \mathfrak{g}^+_4 \) set \( L_T A = T A T^{-1} \) (\( A \in \mathfrak{e}_4 \)). Then \( L_T \) is a linear transformation in \( \mathfrak{e}_4 \), and

**Theorem 5.4.** The mapping \( T \mapsto L_T \) is a 2-to-1 homomorphism of \( \mathfrak{g}^+_4 \) onto \( \mathfrak{e}^+_4 \).

Space-inversion \( P \) and time-reversal \( T \) arise as follows: \( P: A \mapsto J A J^{-1}, T: A \mapsto i A i^{-1} \). Let \( \mathfrak{g} \) be the group of linear transformations in \( \mathbb{E}_4 \) generated by \( \mathfrak{g}^+_4, J, \) and \( i \).

The connection between 2- and 4-component spinors is then contained in

**Theorem 5.5.** The mapping \( \upsilon: \mathbb{S} \mapsto KSK^{-1} \) is an isomorphism of \( \text{SL}(2, \mathbb{C}) \) onto \( \mathfrak{g}^+_4 \) leaving \( \text{SU}(2) \) pointwise fixed.

**Theorem 5.6.** The following diagram is commutative:

\[
\begin{array}{ccc}
\mathbb{W}_4 & \xrightarrow{M_S} & \mathbb{W}_4 \\
\downarrow{\tau} & & \downarrow{\tau} \\
\mathbb{E}_4 & \xrightarrow{L_{KSK^{-1}}} & \mathbb{E}_4
\end{array}
\]

Note that \( \det_4 KSK^{-1} = \det_4 S = |\det_2 S|^2 = 1 \), while \( \det J = \det K = 1 \) and \( \det_4 i = 1 \), whence \( \mathfrak{g}^+_4 \) (or \( \mathfrak{g} \)) and \( \text{SL}(2, \mathbb{C}) \) are subgroups of \( \text{SL}(4, \mathbb{R}) \) whose intersection is \( \text{SU}(2) \).

\( \mathbb{E}_4 \) is also remarkable in that it admits an explicit coordinate-free oriented volume function \( \theta(A_1, A_2, A_3, A_4) = \frac{1}{4} \text{trace}_4 (iA_1A_2A_3A_4J) \), reducing to \((1/2i) \text{trace}_2 (A_1A_2A_3) = (A_1 \times A_2) \cdot A_3 \) when \( A_4 = J \) and \( A_1, A_2, A_3 \) lie in \( \mathbb{G}_3 \) (cf. [3]). Finally, the (Clifford) algebra generated by \( \mathbb{E}_4 \) is just \( B(E_4) \).

**6. The Dirac operator.** Let \( (g_{ij}) = \text{diag}(1, 1, 1, -1) \). Then an ordered orthonormal basis \( (e) \) for \( \mathbb{E}_4 \) is characterized by the identity

\[
e_i e_j + e_j e_i = 2g_{ij}.
\]

Let \( E_4^c \) and \( \mathbb{G}_4^c \) be the respective complexifications of \( E_4 \) and \( \mathbb{E}_4 \) and consider the expression \( \langle Au, v \rangle \), where \( A \) runs over \( \mathbb{E}_4 \) and \( u, v \) run over \( E_4^c \). Since this expression is real linear in \( A \), complex linear in \( u \), and complex antilinear in \( v \), there exists a unique mapping \( F: E_4^c \times E_4^c \rightarrow \mathbb{G}_4^c \) such that

\[
\langle Au, v \rangle = A \cdot F(u, v),
\]
and $F(u, v)$ is complex linear in $u$ and complex antilinear in $v$. In particular, $F(u, Ju)$ lies in $E_4$.

Given now any ordered o.n. basis $e_1, \ldots, e_4$ for $E_4$ consider smooth functions $\psi: E_4 \to E_4'$ and let

$$
(\partial_x \psi)(x) = \lim_{h \to 0} \frac{\psi(x + he_i) - \psi(x)}{h}.
$$

**Definition.** The Dirac operator $\mathcal{D} = e_1 \partial_1 + e_2 \partial_2 + e_3 \partial_3 - e_4 \partial_4$. Then $\mathcal{D}^2 = \partial_1^2 + \partial_2^2 + \partial_3^2 - \partial_4^2$, the d'Alembertian, while the Dirac equation takes the form

$$
\mathcal{D} \psi + \kappa \psi = 0 \quad (\kappa = mc/\hbar),
$$

and the associated charge-current vector $-F(\psi, J\psi)$ satisfies the continuity equation

$$
\text{div } F(\psi, J\psi) = 0.
$$

Finally the relativistic “invariance” properties of the Dirac equation reduce to simple properties of the Dirac operator $\mathcal{D}$.

**Theorem 6.1 (Passive Invariance).** $\langle \mathcal{D} \psi \mid u \rangle = \text{div } F(\psi, u)$ ($u \in E_4'$).

If $T \underline{\in} \mathcal{G}$, let $(\mathcal{D} \mathcal{T}) \psi(x) = \mathcal{T} \psi(L_T^{-1}x) = \mathcal{T} \psi(T^{-1}xT)$.

**Theorem 6.2 (Active Invariance).** $\mathcal{D} \mathcal{T} = \mathcal{T} \mathcal{D}$.

Proofs of the above theorems and some related results will appear elsewhere.

**References**


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