ON THE COUSIN PROBLEMS

BY AVNER FRIEDMAN

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It is well known that if \( \Omega \) is a domain of holomorphy in \( \mathbb{C}^n \) then it is a Cousin I domain; it is also a Cousin II domain if and only if \( H^2(\Omega, \mathcal{Z}) = 0 \). In this work we prove that some general classes of domains which are not domains of holomorphy are both Cousin I and Cousin II domains. Recall that \( \Omega \) is Cousin I (II) if and only if \( H^1(\Omega, 0) = 0 \) (\( H^1(\Omega, 0^*) = 0 \)) where 0 is the sheaf of germs of holomorphic functions under addition and 0* is the sheaf of germs of nowhere-zero holomorphic functions under multiplication. If \( H^1(\Omega, \mathcal{Z}) = 0 \) then "\( \Omega \) Cousin II" implies "\( \Omega \) Cousin I" and if \( H^2(\Omega, \mathcal{Z}) = 0 \) then "\( \Omega \) Cousin I" implies "\( \Omega \) Cousin II."

In what follows we take \( n \geq 3 \) since, for \( n = 2 \), \( \Omega \) is Cousin I if and only if \( \Omega \) is a domain of holomorphy [1].

DEFINITIONS. An open relatively compact set \( A \) in a complex manifold \( X \) is called \( q \)-convex if \( A = \{ z; z \in A_0, \phi(z) < 0 \} \) where \( A_0 \) is a neighborhood of \( \overline{A} \), \( \phi \) is twice continuously differentiable in \( A_0 \), \( \text{grad} \phi \neq 0 \) on \( \partial A \), and the Levi form on \( \partial A \) has at least \( n-q+1 \) positive eigenvalues. If \( A \) and \( B \) are \( q \)-convex, \( B \subset A \), and if there exists a function \( \phi(z, t) (z \in A_0, 0 \leq t \leq 1) \) twice continuously differentiable in \( z \) such that the sets \( D_t = \{ z; z \in A_0, \phi(z, t) < 0 \} \) are \( q \)-convex and lie in \( A_0 \) and \( D_0 = A, D_1 = B \), then we say that \( A \) and \( B \) are \( q \)-convex homotopic. Example: if \( A, B \) are strictly convex then they are 1-convex homotopic.

Let \( K_1, L_1 \) be open convex sets in the \( z_1 \)-plane, \( 0 \in L_1, \overline{L_1} \subset K_1 \), and set \( A_1 = K_1 \setminus \overline{L_1} \). Let \( K' = K_2 \times \cdots \times K_n, L' = L_2 \times \cdots \times L_n \) be open convex generalized polydiscs \( (K_j, L_j \text{ lie in the } z_j \text{-plane}) \) with \( 0 \in L' \), \( \overline{L'} \subset K' \). All the previous sets are taken to be bounded. Set \( G_0 = A_1 \times K', G_1 = K_1 \times (K' \setminus \overline{L'}) \), \( G = G_0 \cup G_1 \).

**LEMMA 1.** \( G \) is both Cousin I and Cousin II.

The proof that \( G \) is Cousin I is a straightforward generalization of the proof of [7, Hilfsatz]. Thus, it remains to show that \( H^2(G, \mathcal{Z}) = 0 \).

**LEMMA 2.** \( H^r(G, \mathcal{Z}) = 0 \) for \( 0 < r \leq 2n \).

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PROOF. Moving the points of $K_1 \setminus L_1$ and of $K' \setminus L'$ inward along the rays from 0 we get a strong deformation retract of $G$ into a set homeomorphic to $N = S^m \times D^{m+1} \cup D^2 \times S^m$ where $m = 2n - 1$, $S^m$ and $S^2$ are unit $p$-spheres, and $D^2$, $D^m$ are unit $p$-balls. The join $S^1 \circ S^m$ can be described by $x \cos t + y \sin t$, $0 \leq t \leq \pi/2$, where $x \in S^1$, $y \in S^m$. Introduce the function $(\theta, \phi, t) \mapsto ((1, \theta), (4t/\pi, \phi))$ which maps homeomorphically the set $J_-$ in $S^1 \circ S^m$ corresponding to $0 \leq t \leq \pi/4$ onto $S^2 \times D^{m+1}$, where $\theta, \phi$ are the angular coordinates of $x, y$. Similarly we introduce a map of $J_+$ (for which $\pi/4 \leq t \leq \pi/2$) onto $D^2 \times S^m$. Thus $N$ is homeomorphic to $S^1 \circ S^m$, and since the latter is known to be homeomorphic to $S^{m+2}$, the result follows.

LEMMA 3. The envelope of holomorphy of $G$ contains $K_1 \times K'$.

PROOF. Given $f$ holomorphic in $G$, for any $\xi \in A_1 \times K'$ we represent $f(\xi)$ by Cauchy's formula, where the $\sigma_1$-contour is composed of one part lying near $\partial K_1$ and another "inner" part, say $J$, lying near $\partial L_1$, and where the $\sigma_j$-contour, for $j \geq 2$, is in $K_j \setminus L_j$. Now notice that the integral over $J$ vanishes.

LEMMA 4. Let $X$ be an open set in a complex manifold and let $A, B$ be subsets of $X$ $n$-convex homotopic, and $B \subseteq A$. Then any two Cousin II data in $X \setminus B$ which are equivalent in $X \setminus A$ are also equivalent in $X \setminus B$.

The proof is a rather obvious extension of [5, Satz 1, A1, A2] provided one employs a theorem of Lewy [4] (see also [3]) concerning local analytic continuation across the boundary of each $\partial D_i$.

LEMMA 5. Let $X$ be an open set in $C^n$ and let $L = L_1 \times \cdots \times L_n$ be a generalized polydisc which is open, convex and bounded, and $\overline{L} \subset X$. Then any Cousin II data $(g_P, U_P)$ in $X \setminus \overline{L}$ can be continued into $X$.

PROOF. Let $K = K_1 \times \cdots \times K_n$ be an open convex bounded generalized polydisc with $\overline{L} \subset K$, $\overline{K} \subset X$ and introduce $G$ as in Lemma 1. Clearly $G \subset X \setminus \overline{L}$. Since $G$ is Cousin II, there exists an $f$ holomorphic in $G$ such that $(f, G)$ is equivalent (in $G$) to the given Cousin data. Continue $f$ to $K_1 \times K'$ (by Lemma 3). For each $P$ in $(K_1 \times K') \cap G$ we take the germ $f_P$ of $f$ in the neighborhood $G$ of $P$. For $P$ in $(K_1 \times K') \setminus G$ we take a sufficiently small neighborhood $V_P$ of $P$ such that its intersection with $X \setminus \overline{L}$ lies in $G$, and then take $f_P$ to be the germ of the continuation of $f$. We have thus continued the Cousin data into $X$.

LEMMA 6. Lemma 5 remains true if $L$ is any open, strictly convex and bounded set with $C^2$ boundary, and $\overline{L} \subset X$.

\footnote{I am indebted to Daniel Kahn for the proofs of this lemma and of Lemma 7.}
PROOF. Let $R \subseteq \partial L$. We first wish to continue the data to a neighborhood $W$ of $R$. Assume that $\text{Re}(z_1) = 0$ is the tangent hyperplane to $L$ at $R$, that $L$ lies in $\text{Re}(z_2) < 0$, and that $R$ is at the origin.

We apply a modified version of Lemmas 1–3 where $G$ is defined differently, namely, $A_1 = \{ z_1 ; 0 < \text{Re}(z_1) < \alpha , \mid \text{Im}(z_1) \mid < \beta \}$, $K_1 = \{ z_1 ; 0 < \text{Re}(z_1) < \alpha , \mid \text{Im}(z_1) \mid < \beta \}$, and follow the argument of Lemma 5. We then need to show that the data obtained by the continuation of $f$ agree with the given data in $(X \setminus L) \cap W$. This is done by extending the argument 3 of [5, p. 345]. However that argument is erroneous (since the existence of a smallest $t^*$ is not justified). Instead we construct a family of $C^2$ hypersurfaces $S(t)$ $(1 \leq t \leq 2)$ with boundary in $(K_1 \setminus A_1) \times (K \setminus L')$ such that $S(t)$, at each of its points, is convex in at least one tangential direction (in fact, we can take it convex in $2n - 1$ independent directions), $S(t)$ lies outside $L$ if $t > 1$, $S(1) \supset (X \setminus L) \cap W$, and $S(2) \subset G$. Then, by the proof of Lemma 4, we show that the set of $t$'s such that at all the points of $S(t)$ $(t < t \leq 2)$ the two sets of data are equivalent, is both open and closed. Having continued the data to $W$, the argument $C$ of [5, p. 343], combined with Lemma 4, completes the proof.

DEFINITION. An $n$-convex subset $A$ of $X \subset \mathbb{C}^n$ is said to have the property $(P)$ if it is $n$-convex homotopic to a set $B \supset A$, and if there exists a convex set $L$ such that $A \subset L \subset \mathbb{C} \subset B$.

THEOREM 1. Let $X$ be an open set in $\mathbb{C}^n$ and let $A$ be an $n$-convex subset of $X$ satisfying the property $(P)$. Then any Cousin II data in $X \setminus A$ can be continued into $X$.

PROOF. Consider the given data $V$ restricted to $X \setminus L$. By Lemma 5 there exists a continuation $V'$ of the data to $X$. Since $V$, $V'$ are equivalent in $X \setminus B$, they are also equivalent in $X \setminus A$ (by Lemma 4).

Rothstein [5, Satz 1*] stated a similar theorem for $n = 3$, replacing “$n$-convex” by “analytic polyhedron” and omitting the condition $(P)$, but in his proof 2 there occurs a serious mistake. The same remark applied to his treatment of the first Cousin problem in [6].

From Theorem 1 we get:

THEOREM 2. Let $X$ be a Cousin II domain in $\mathbb{C}^n$ and let $A$ be an $n$-convex subset of $X$ having the property $(P)$. Then $X \setminus A$ is a Cousin II domain.

LEMMA 7. Let $X$ be an open set on a real $n$-dimensional differential manifold satisfying $H^q(X, \mathbb{Z}) = 0$ for $q = 1, \ldots , m$ $(m < n)$, and let $A$ be a contractible relatively compact subset of $X$ with continuously differentiable boundary. Then $H^q(X \setminus A, \mathbb{Z}) = 0$ for $q = 1, \ldots , m$.  

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Proof. Write $H^r(N)$ for $H^r(N, \mathbb{Z})$. Since $\partial A$ is differentiable, $X \setminus \overline{A}$ can be deformed continuously to an open set $B$ which contains $\partial A$. We have $H^r(X \setminus \overline{A}) = H^r(B)$ for $r \geq 0$. Since $A$ is contractible, $H^r(A) = 0$ if $r > 0$. Next, $A \cap B$ can be deformed continuously to $\partial A$ and, therefore, $H^r(A \cap B) = H^r(\partial A)$. By Lefschetz Duality Theorem [2], $H^r(A, \partial A) = 0$ if $0 \leq r < n$, and from the exact sequence $H^r(A) \rightarrow H^r(\partial A) \rightarrow H^r(A, \partial A)$ we then infer that $H^r(\partial A) = 0$; hence $H^r(A \cap B) = 0$ if $0 < r < n$. Noting that $A \cap B \neq \emptyset$, we can write down the Mayer-Vietoris exact sequence $H^r(X) \rightarrow H^r(A) \oplus H^r(B) \rightarrow H^r(A \cap B)$ and obtain $H^r(B) = 0$ if $1 \leq r \leq m$.

Theorem 3. Let $X$ be a Cousin I domain in $\mathbb{C}^n$ and let $A$ be an $n$-convex subset of $X$ having the property (P). If $H^q(X, \mathbb{Z}) = 0$ for $q = 1, 2$ and if $A$ is contractible, then $X \setminus \overline{A}$ is a Cousin I domain.

Indeed, $X$ is Cousin II and, by Theorem 2, also $X \setminus \overline{A}$ is Cousin II. Since, by Lemma 7, $H^r(X \setminus \overline{A}, \mathbb{Z}) = 0$, $X \setminus \overline{A}$ is Cousin I.

Corollary. If $X$ is a domain of holomorphy in $\mathbb{C}^n$, if $H^q(X, \mathbb{Z}) = 0$ for $q = 1, 2$, and if $A$ is as in Theorem 3, then $X \setminus \overline{A}$ is both Cousin I and Cousin II.

Theorems 1–3 extend to the case where instead of one hole $A$ there is a finite number of holes. The results also extend to sets $X$ on complex manifolds, provided $B$ (in (P)) lies in one coordinate patch.

Added in proof. (I) Define “real $2g$-convex homotopic” analogously to “$g$-convex homotopic” by requiring the manifolds to be strictly convex in at least $n - q + 1$ complex directions of the tangent hyperplanes. Theorems 1–3 remain true if the condition (P) is relaxed by taking $L$ to be real $2(n - 1)$-convex homotopic to a point. Indeed, modify the proof of Lemma 6 ($L$ is strictly convex in $z_2, z_3$ directions) taking $K_j = L_j$ for $j = 4, \ldots, n$ and, in the definition of $A_1$, $\epsilon < \text{Re}(z_1) < \delta$. For fixed $\xi \in W, \zeta \in L$, take $S(t)$ to be 5-dimensional surfaces lying outside $L$, with $z_j = \zeta_j$ ($j = 4, \ldots, n), \xi \in S(t)$ for some $t > 1$, such that they are $1$-convex. To construct $S(t)$ take $E_j$ ($j = 1, 2$) convex 4-dimensional surfaces on $\text{Re}(z_i) = -\beta_j$ ($\beta_1 < \beta_2$) and in $G$, and take 4-dimensional surface $F$, on $\text{Re}(z_i) = -\beta_1$, lying outside $L$ such that its intersection $F_\alpha$ with $\text{Im}(z_1) = \alpha$ is convex for all small $\alpha$. Let $E^* \subset E \cap \{ \text{Im}(z_1) = \alpha \}$ and take $R^* = (\gamma, 0, \ldots, 0) (\gamma > 0)$ outside $L$. $S(2)$ is a convex cap with top $R^*$, base $E^*$, and passing through $E_1$. Deform $E_1$ into $F$ (by deforming $E^*_\alpha$ into $F_\alpha$) and, correspondingly, deform the “meridians” issuing from $R^*$ to $E^*$. $S(t)$ is the deformation of $S(2)$ at stage $t$ ($1 \leq t < 2$).
(II) In [Math. Ann. 120 (1955), 96–138] Rothstein gave a proof of Theorem 1 with “$n$-convex” replaced by “$(n-1)$-convex” and with “(P)” replaced by “$A$ is a star domain.” His proof applies also to continuation of analytic sets, but our proof is much simpler.

**References**


Northwestern University