SOLVABILITY OF THE FIRST COUSIN PROBLEM AND VANISHING OF HIGHER COHOMOLOGY GROUPS FOR DOMAINS WHICH ARE NOT DOMAINS OF HOLOMORPHY

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Communicated by F. Browder, June 7, 1965

This work is a sequel to [1]: In [1] we considered the vanishing of the first cohomology groups with coefficients in \(0, 0^*\) for sets \(X\setminus A\) whereas in the present work we consider the same question for higher cohomology; at the same time we obtain some additional results for the first Cousin problem. As in [1] we take \(n \geq 3\).

Scheja [3] proved that if \(X\) is an open set in \(C^n\) and \(A\) is an analytic closed subset of \(X\) of dimension \(\leq n-q-2\), then the natural homomorphism

\[
H^q(X, 0) \rightarrow H^q(X\setminus A, 0)
\]

is bijective. We shall prove:

**Theorem 1.** Let \(A\) be a closed bounded generalized polydisc in an open set \(X\) of \(C^n\). Then the natural homomorphism (1) is bijective for any \(1 \leq q \leq n-2\).

**Proof.** Set \(A = L_1 \times \cdots \times L_n\) and let \(K = K_1 \times \cdots \times K_n\) be an open generalized polydisc with \(A \subset K \subset \overline{K} \subset X\). Set \(L' = L_2 \times \cdots \times L_n,\ K' = K_2 \times \cdots \times K_n,\ G_0 = (K_1 \setminus L_1) \times K',\ G_1 = K_1 \times (K' \setminus L')\), \(G = G_0 \cup G_1\). By a straightforward generalization of [3, Hilfsatz] one gets \(H^q(G, 0) = 0\). We now introduce a covering \(U = \{ U_i \}\) of \(X\setminus A\) where all the \(U_i\) are domains with \(H^q(U_i, 0) = 0\) and precisely \(q+1\) of them, say \(U_{i_0}, \cdots, U_{i_q}\), coincide with \(G\). By Leray’s theorem [2], the canonical homomorphism

\[
H^q(N(U), 0) \rightarrow H^q(X\setminus A, 0)
\]

(where \(N(U)\) is the nerve of \(U\)) is bijective.

We next introduce a covering \(U' = \{ U'_i \}\) of \(X\) where \(U'_{i_0} = \cdots = U'_{i_q} = K_1 \times K'\) and \(U'_i = U_i\) for all other indices \(i\). Again, the canonical map

\[
H^q(N(U'), 0) \rightarrow H^q(X, 0)
\]

is bijective.

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1 This work was partially supported by the Alfred P. Sloan Foundation and by the NASA Grant NGR 14-007-021.
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is bijective. We shall now construct a map

\[(4) \quad H^q(N(U), \emptyset) \rightarrow H^q(N(U'), \emptyset).\]

Let \(f \in H^q(N(U), \emptyset)\). We may view it as a \(q\)-cocycle. Let \(f_{i_0}, \ldots, f_{i_q}\) be the section of \(f\) on \(U_{i_0} \cap \cdots \cap U_{i_q} = G\). The proof of Lemma 3 in [1] can be extended to show that \(f_{i_0}, \ldots, f_{i_q}\) can be continued analytically to \(K_1 \times K'\). The continued function \(f'_{i_0}, \ldots, f'_{i_q}\) thus obtained is defined on \(U'_{i_0} \cap \cdots \cap U'_{i_q}\). We now define \(f'_{i_0}, \ldots, f'_{i_q}\) for any set of distinct indices \(\{j_0, \ldots, j_q\}\) which does not coincide with the set \(\{i_0, \ldots, i_q\}\).

Since among the \(j_k\)'s there is at least one index, say \(i\), with \(i \neq i_k\) for all \(0 \leq k \leq q\), and, consequently, \(U'_i = U_i \subset X \setminus A\), we have \(U'_i \cap (K_1 \times K') = U'_i \cap G\). Hence \(U'_{j_0} \cap \cdots \cap U'_{j_q} = U_{j_0} \cap \cdots \cap U_{j_q}\), and we can take \(f'_{j_0}, \ldots, f'_{j_q} = f_{j_0}, \ldots, f_{j_q}\).

We have thus defined a \(q\)-cochain \(f'\) on \(N(U')\). \(f'\) is cocycle. Indeed, observing that \(U'_{i_0} \cap \cdots \cap U'_{i_q+1}\) coincides with \(U_{j_0} \cap \cdots \cap U_{j_q+1}\) if all the \(j_k\) are distinct from each other, and that the analytic function \(f'_{i_0}, \ldots, f'_{i_q}\) restricted to either of these sets coincides with \(f_{j_0}, \ldots, f_{j_q}\), the equation \(\delta f' = 0\) implies \(\delta f = 0\).

We next show that if \(f = \delta g\) then there is a \((q-1)\)-chain \(g'\) with \(\delta g' = f'\). If (a) \(\{j_0, \ldots, j_{q-1}\} \subset \{i_0, \ldots, i_q\}\) then we take \(g'_{i_0}, \ldots, g'_{i_{q-1}}\) to be the analytic continuation of \(g_{i_0}, \ldots, g_{i_{q-1}}\) to \(U'_{i_0} \cap \cdots \cap U'_{i_{q-1}}\), whereas if (a) does not hold then \(U'_{j_0} \cap \cdots \cap U'_{j_{q-1}} = U_{j_0} \cap \cdots \cap U_{j_{q-1}}\) and we take \(g'_{j_0}, \ldots, g'_{j_{q-1}} = g_{j_0}, \ldots, g_{j_{q-1}}\). With \(g'\) thus constructed, the relation \(\delta g' = f'\) over \(U'_{j_0} \cap \cdots \cap U_{j_{q-1}}\) in case (b) \(\{j_0, \ldots, j_q\} = \{i_0, \ldots, i_q\}\) follows from the relation \(\delta g = f\) over \(U_{j_0} \cap \cdots \cap U_{j_q}\) by analytic continuation, whereas in case (b) does not hold it coincides with the relation \(\delta g = f\) over \(U_{j_0} \cap \cdots \cap U_{j_q}\).

We have thus shown that the map \(f \rightarrow f'\) defines a homomorphism (4). This map is surjective since, given \(f'\), its restriction \(f\) to \(N(U)\) is mapped into \(f'\) by the above map. It is also injective since if \(f' = \delta g'\) for some \((q-1)\)-cochain \(g'\) over \(N(U')\), then the restriction \(g\) of \(g'\) to \(N(U)\) clearly satisfies \(f = \delta g\). Noting finally that the map \(f \rightarrow f'\) is the inverse of the restriction map, and combining (2)-(4), (1) follows.

COROLLARY. If \(H^q(X, \emptyset) = 0\) then \(H^q(X \setminus A, \emptyset) = 0\). In particular, if \(X\) is Cousin I then \(X \setminus A\) is Cousin I.

THEOREM 2. Let \(A, B\) be two closed bounded subsets of an open set \(X \subset \mathbb{C}^n\) and let \(P\) be a closed generalized polydisc with \(A \subset \text{int } P \subset P \subset \text{int } B\). If, for some \(1 \leq q \leq n-2\), the natural homomorphism

\[(5) \quad H^q(X \setminus A, \emptyset) \rightarrow H^q(X \setminus B, \emptyset)\]

is injection, then there exists a homomorphism \(\lambda: H^q(X \setminus A, \emptyset) \rightarrow H^q(X, \emptyset)\)
such that πλ = identity, where π is the map (1) (and, consequently, π is surjective); in particular, if $H^q(X, \theta) = 0$ then $H^q(X \setminus A, \theta) = 0$.

**Proof.** Take coverings $U^1, U^2, U^3, U^4$ of $X, X \setminus A, X \setminus P, X \setminus B$ respectively whose open sets are domains of holomorphy and such that the sets of $U^i$ ($i = 2, 3, 4$) are among the sets of $U^{i-1}$. Given $f_2 \in H^q(N(U^3), \theta)$ there corresponds to it (by restriction) a unique element $f_4$ in $H^q(N(U^4), \theta)$ and a unique element $f_3$ in $H^q(N(U^3), \theta)$; $f_4$ is the restriction of $f_3$. By Theorem 1 there exists an $f_1 \in H^q(N(U^1), \theta)$ whose restriction to $N(U^3)$ is $f_2$. Hence the restriction of $f_1$ to $N(U^4)$ is $f_4$. Since $f_1$ and $f_2$ have the same restriction on $N(U^4)$, the injectivity of (5) implies that the restriction of $f_1$ to $N(U^3)$ is $f_2$. Thus the map $f_2 \mapsto f_1$ is an inverse of the restriction map $H^q(N(U^1), \theta) \rightarrow H^q(N(U^3), \theta)$. The assertion of the theorem now follows with $X$ being the image of the homomorphism $f_2 \mapsto f_1$ under the canonical map corresponding to $H^q(N(U^3), \theta) \rightarrow H^q(X \setminus A, \theta), H^q(N(U^1), \theta) \rightarrow H^q(X, \theta)$.

**Generalizations.** By successive applications of Theorem 1 we get:

1. If $A_1, \ldots, A_m$ are closed bounded generalized polydiscs such that $A_j \cap A_k = \emptyset$ if $j \neq k$, then the natural map $H^q(X, \theta) \rightarrow H^q \left( X \setminus \left( \bigcup_{i=1}^m A_i \right), \theta \right)$ is bijective.

2. (2) Theorem 1 extends to the case where $X$ is an open set on a complex manifold provided $A$ is contained in one coordinate patch and its image in $C^n$ is a generalized polydisc. Theorem 2 and (1) have similar extensions.

By slightly modifying the proof of Theorem 1 we obtain:

3. If $X = X_1 \times K_{p+1} \times \cdots \times K_2, A = A_1 \times K_{p+1} \times \cdots \times K_n$ where $X_1$ is any open set of $C^n$ and $K_j$ is an open set in the $z_j$-plane, then the homomorphism (1) is bijective if $1 \leq q \leq p - 2$.

4. If $A$ in Theorem 1 is convex, then (see [1]) $H^q(G, \theta^*) = 0$. By modifying the proof of Theorem 1 we find that the natural homomorphism $H^q(X, \theta^*) \rightarrow H^q(X \setminus A, \theta^*)$ is bijective. The analogs of Theorem 2 and (1)–(3) are also valid.

We shall now give a different approach to proving results similar to Theorem 1. Since this approach does not yield a result as general as Theorem 1, we shall only sketch it. Let $X = K_1 \times \cdots \times K_n, A = L_1 \times \cdots \times L_n$ be generalized polydiscs. We say that the condition $(A_m)$ holds if for each $j = 1, \ldots, m$ either (a) $K_1$ is the whole
plane $C$ and then $L_j$ is an arbitrary closed bounded subset of $K_j$, or (b) $K_j = C$ and then $L_j$ consists of a finite number of points. The $L_j$ for $j = m+1, \ldots, n$ are arbitrary closed subsets of $K_j$.

**Theorem 3.** If $(A_m)$ holds for some $2 \leq m \leq n$ then $H^q(X \setminus A, \partial) = 0$ for $1 \leq q \leq \min(m - 1, n - 2)$. The relations $H^{n-1}(X \setminus A, \partial) \neq 0$, $H^q(X \setminus A, \partial) = 0$ for $q \geq n$ are valid under the assumption $(A_0)$.

**Proof.** Setting $\Delta_j = K_1 \times \cdots \times K_{j-1} \times (K_j \setminus L_j) \times K_{j+1} \times \cdots \times K_n$ and noting that $H^q(\Delta_j, \partial) = 0$ for $q \geq 1$, it suffices to consider $H^q(U, \partial)$, where $U = \{\Delta_1, \ldots, \Delta_n\}$. We consider only the case $1 \leq q \leq n - 2$. Denote by $I_{i_1, \ldots, i_h}(h)$ the Cauchy integral of $h$ with the $i$th contour being $\partial K_i$ if $i \neq j$ for all $p$, and $\partial L_i$ if $i = j_p$ for some $p$. (Actually one should replace $\partial K_m, \partial L_m$ by smooth $\partial K_m, \partial L_m$ which approximate $\partial K_m, \partial L_m$.) Then we can represent each component $f_{i_0, \ldots, i_q}$ of a $q$-cochain $f$ by

$$f_{i_0, \ldots, i_q} = \sum_{h=0}^{q+1} \sum_{i_1, \ldots, i_h} I_{i_1, \ldots, i_h}(f_{i_0, \ldots, i_q}).$$

**Lemma 1.** Consider a domain $D = K \setminus L$ in the complex plane, where $K$ is the whole plane and $L$ is any closed bounded set with $C^1$ boundary $\partial L$. Let $\phi(z)$ be any analytic function in $D$ and let $\psi(z)$ be any continuous function on $\partial L$ such that

$$\int_{|t| = R} \frac{\phi(t)}{t - z} \, dt + \int_{\partial L} \frac{\psi(t)}{t - z} \, dt = 0 \text{ in } D \cap \{z; |z| < R\}$$

for all $R$ sufficiently large. Then, for all $R$ sufficiently large,

$$\int_{|t| = R} \frac{\phi(t)}{t - z} \, dt = \int_{\partial L} \frac{\psi(t)}{t - z} \, dt = 0 \text{ in } D \cap \{z; |z| < R\}.$$

A similar result holds in case $K$ is a bounded set with $C^1$ boundary and $L$ consists of a finite number of points. Using these results, the condition $\delta f = 0$ implies the following system of equations:

If $i_0 < \cdots < i_h \leq m < i_{h+1} < \cdots < i_{q+1}$ for some $0 \leq h \leq q + 1$, and if $i_{j_1} < \cdots < i_{j_k} \leq m$ for some $0 \leq k \leq h$, then

$$f_{i_0, \ldots, i_q} = \sum_{\alpha_1, \ldots, \alpha_p} \left( \sum_{\nu=0}^{q+1} (-1)^\nu I_{i_1, \ldots, i_h, \alpha_1, \ldots, \alpha_p} \prod_{\nu=0}^{q+1} \right) = 0,$$

where in the third summation $\nu \neq j_1, \ldots, \nu \neq j_k$ and $\nu \neq \lambda_1, \ldots, \nu \neq \lambda_p$.

To find $g$ satisfying $\delta g = f$, we try to represent $g_{i_0, \ldots, i_{q-1}}$ analogously to (6), and then the relation $\delta g = f$ is a consequence of the following system of equations:
If \( i_0 < \cdots < i_{h-1} \leq m < i_h < \cdots < i_q \) for some \( 0 \leq h - 1 \leq q \), and if \( i_{j_1} < \cdots < i_{j_k} \leq m \) for some \( 0 \leq k \leq h - 1 \), then

\[
\sum_{p=0}^{q-h+1} \sum_{h_1 < \cdots < h_p} I_{i_{j_1}} \cdots \cdots I_{i_{j_k}} \cdots \cdots I_{i_q} \left( \sum_{r=0}^{q} (-1)^r g_{i_0} \cdots \cdots g_{i_q} \right) = 0,
\]

where in the third summation of the first term \( \nu \neq j_1, \cdots, \nu \neq j_k \) and \( \nu \neq \lambda_1, \cdots, \nu = \lambda_p \).

Using (7) we can solve (8) as follows: If \( i_0 > 1 \), or if \( i_0 = 1, i_{j_1} > 1 \) then \( g_{i_0} \cdots \cdots g_{i_{q-1}} = f_{i_0} \cdots \cdots f_{i_{q-1}} \). If \( i_0 = i_{j_1} = 1 \) and if \( i_1 > 2 \) or \( i_1 = 2, i_{j_2} > 2 \) then \( g_{i_0} \cdots \cdots g_{i_{q-1}} = f_{i_0} \cdots \cdots f_{i_{q-1}} \). We proceed in this manner and finally define, in case \( i_0 = i_{j_1} = 1, \cdots, i_{k-1} = i_{j_k} = k, g_{i_0} \cdots \cdots g_{i_{q-1}} = f_{i_{k+1}, i_0} \cdots \cdots i_{q-1} \).

This method extends also to the situations described in (1), (3) above.

*Added in proof.* The relation \( H^{\nu-2}(X \setminus A, \emptyset) \neq 0 \) holds if in (3) \( X_1 \) and \( A_1 \) are both generalized polydiscs. Taking \( \Omega_m = X_m \setminus A_m \) where \( X_m, A_m \) are generalized polydiscs with \( X_m \not\subseteq X, A \not\subseteq A \) one derives, for fixed \( 1 \leq q \leq n - 2 \), examples of domains \( \Omega_m \) with \( \Omega_{m-1} \cap \Omega_m \), such that \( H^r(\Omega, \emptyset) = 0 \) for \( 1 \leq r \leq n - 2 \) but \( H^r(\Omega_m, \emptyset) \neq 0 \) where \( \Omega = \text{int} (\lim \Omega_m) \).

By Dolbeault’s theorem, \( H^r(\Omega, \emptyset) = 0 \) if and only if for any \( C^\infty(\Omega) \) form \( f \) of bidegree \((0, q)\) with \( \bar{\partial} f = 0 \) there is a \( C^\infty(\Omega) \) form \( u \) with \( \bar{\partial} u = f \). By modifying the proof in [2, p. 29] we find: If for some \( q > 1, \Omega_m \subset \Omega = \lim \Omega_m, H^r(\Omega_m, \emptyset) = 0 \) for \( r = q - 1, q \), then \( H^q(\Omega_m, \emptyset) = 0 \). Also if \( H^1(\Omega_m, \emptyset) = 0 \) and if for any \( u \) holomorphic in \( \Omega_m \) and \( \epsilon > 0 \) there is a \( v \) holomorphic in \( \Omega_{m+1} \) with \( |u - v| < \epsilon \) in \( \Omega_{m-1} \), then \( H^1(\Omega, \emptyset) = 0 \); this can be applied to \( \Omega_m = X_m \setminus A_m \) as in [1, Theorem 3].

**References**


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