RADON-FOURIER TRANSFORMS ON SYMMETRIC SPACES AND RELATED GROUP REPRESENTATIONS

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In §2 we announce some results in continuation of [10], connected with the Radon transform. §1 deals with tools which also apply to more general questions and §§2–3 contain some applications to group representations. A more detailed exposition of §2 appears in Proceedings of the U. S.-Japan Seminar in Differential Geometry, Kyoto, June, 1965.

1. Radial components of differential operators. Let \( V \) be a manifold, \( v \) a point in \( V \) and \( V_v \) the tangent space to \( V \) at \( v \). Let \( G \) be a Lie transformation group of \( V \). A \( C^\infty \) function \( f \) on an open subset of \( V \) is called locally invariant if \( Xf = 0 \) for each vector field \( X \) on \( V \) induced by the action of \( G \).

Suppose now \( W \) is a submanifold of \( V \) satisfying the following transversality condition:

\[(T) \quad \text{For each } w \in W, V_w = W_w + (G \cdot w)_w \quad (\text{direct sum}).\]

If \( f \) is a function on a subset of \( V \) its restriction to \( W \) will be denoted \( f|_W \).

**Lemma 1.1.** Let \( D \) be a differential operator on \( V \). Then there exists a unique differential operator \( \Delta(D) \) on \( W \) such that

\[(Df)|_W = \Delta(D)f|_W \]

for each locally invariant \( f \).

The operator \( \Delta(D) \) is called the radial component of \( D \). Many special cases have been considered (see e.g. [1, §2], [4, §5], [5, §3], [7, §7], [8, Chapter IV, §§3–5]).

Suppose now \( dw \) (resp. \( dw \)) is a positive measure on \( V \) (resp. \( W \)) which on any coordinate neighborhood is a nonzero multiple of the Lebesgue measure. Assume \( dg \) is a bi-invariant Haar measure on \( G \).

Given \( u \in C_c^\infty(G \times W) \) there exists [7, Theorem 1] a unique \( f_u \in C_c^\infty(G \cdot W) \) such that

\[ \int_{G \times W} F(g \cdot w) u(g, w) \, dg \, dw = \int_V F(v) f_u(v) \, dv \quad (F \in C_c^\infty(G \cdot W)). \]

Let \( \phi_u \in C_c^\infty(W) \) denote the function \( w \mapsto f_u(g, w) \) \( dg \).

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Theorem 1.2. Suppose $G$ leaves $dv$ invariant. Let $T$ be a $G$-invariant distribution on $G \cdot W$. Then there exists a unique distribution $\bar{T}$ on $W$ such that

$$\bar{T}(\phi_u) = T(f_u), \quad u \in C_0^\infty(G \times W).$$

If $D$ is a $G$-invariant differential operator on $V$ then

$$(DT)^{-} = \Delta(D)\bar{T}.$$ 

The proof is partly suggested by the special case considered in [7, §9]. See also [12, §4].

2. The Radon transform and conical distributions. Let $G$ be a connected semisimple Lie group, assumed imbedded in its simply connected complexification. Let $K$ be a maximal compact subgroup of $G$ and $X$ the symmetric space $G/K$. Let $G = KAN$ be an Iwasawa decomposition of $G$ ($A$ abelian, $N$ nilpotent) and let $M$ and $M'$, respectively, denote the centralizer and normalizer of $A$ in $K$. The space $\Xi$ of all horocycles $\xi$ in $X$ can be identified with $G/MN$ [10, §3]. Let $D(X)$ and $D(\Xi)$ denote the algebras of $G$-invariant differential operators on $X$ and $\Xi$, respectively; let $S(A)$ denote the symmetric algebra over the vector space $A$ and $I(A)$ the set of elements in $S(A)$ which are invariant under the Weyl group $W = M'/M$. There are isomorphisms $\Gamma$ of $D(X)$ onto $I(A)$ [6, p. 260], [9, p. 432] and $\Gamma$ of $D(\Xi)$ onto $S(A)$ [10, p. 676].

The Radon transform $f \mapsto f^\phi$ ($f \in C_c^\infty(X)$) and its dual $\phi \mapsto \phi^x$ ($\phi \in C^\infty(\Xi)$) are defined by

$$f^\phi(\xi) = \int_\xi f(x) dm(x), \quad \phi^x(\xi) = \int_\xi \phi(\xi) d\mu(\xi) \quad (x \in X, \xi \in \Xi)$$

where $dm$ is the measure on $\xi$ induced by the canonical Riemannian structure of $X$, $\bar{x}$ is the set of horocycles passing through $x$ and $d\mu$ is the measure on $\bar{x}$ invariant under the isotropy subgroup of $G$ at $x$, satisfying $\mu(\bar{x}) = 1$. The easily proved relation

$$\int_X f(x)\phi^x(dx = \int_\Xi f(\xi)\phi(\xi)d\xi \quad (f \in C^\infty_c(X), \phi \in C^\infty(\Xi))$$

$d\xi$ and $d\xi$ being $G$-invariant measures on $X$ and $\Xi$, respectively, suggests immediately how to extend the integral transforms above to distributions.

Let $\mathfrak{g}$ and $\mathfrak{a}$ be the Lie algebras of $G$ and $A$, respectively, and $\mathfrak{a}^*$
the dual space of \( \mathfrak{A} \). Let \( \lambda \mapsto c(\lambda) \) be the function on \( \mathfrak{A}^\ast \) giving the Plancherel measure \( |c(\lambda)|^{-2}d\lambda \) for the \( K \)-invariant functions on \( X \) (Harish-Chandra [6, p. 612]). Let \( j \) be the operator on rapidly decreasing functions on \( A \) which under the Fourier transform on \( A \) corresponds to multiplication by \( c^{-1} \). Let \( \rho \) denote the sum (with multiplicity) of the restricted roots on \( \mathfrak{A} \) which are positive in the ordering given by \( N \). Let \( \varphi \) denote the function on \( \mathfrak{Z} \) defined by

\[
e^{\varphi(kaMN)} = \exp[\rho(\log a)] \quad (k \in K, a \in A).
\]

Viewing \( \mathfrak{Z} \) as a fibre bundle with base \( K/M \), fibre \( A \) [10, p. 675] we define the operator \( \Lambda \) on suitable functions \( \phi \) on \( \mathfrak{Z} \) by

\[
\Lambda \phi = \int (\phi \circ j)(\varphi) |F|
\]

where \( |F| \) denotes restriction to any fibre \( F \). Similarly, the complex conjugate of \( c^{-1} \) determines an operator \( \Lambda^\ast \). By means of the Plancherel formula mentioned one proves (cf. [11, §6]).

**Theorem 2.1.** There exist constants \( c, c' > 0 \) such that

\[
\int_X |f(x)|^2dx = c' \int_\mathfrak{Z} |\Lambda f(\xi)|^2d\xi, 
\]

\[
f = c(\Lambda \Lambda^\ast)^{-1}
\]

for all \( f \in C_0^\infty(X) \).

If all Cartan subgroups of \( G \) are conjugate, the operators \( j \) and \( \Lambda \) are differential operators (\( c^{-1} \) is a polynomial). Considering \( jj \) is an element in \( I(A) \) we put \( \Box = \Gamma^{-1}(jj) \in D(X) \). Then (3) can be written in the form

\[
f = c \Box ((\Lambda \Lambda^\ast)^{-1}), \quad f \in C_0^\infty(X),
\]

which is more convenient for applications [10, §7]. For the case when \( G \) is complex a formula closely related to (3) was given by Gelfand-Graev [2, §5.5].

Let \( x_0 \) and \( \xi_0 \) denote the origins in \( X \) and \( \mathfrak{Z} \), respectively. The space \( B = K/M \) can be viewed as the set of Weyl chambers emanating from \( x_0 \) in \( X \). If \( \xi = ka \cdot \xi_0 \) \((k \in K, a \in A)\) we say that the Weyl chamber \( kM \) is normal to \( \xi \) and that \( a \) is the complex distance from \( x_0 \) to \( \xi \). If \( x \in X, b \in B \) let \( \xi(x, b) \) be the horocycle with normal \( b \) passing through \( x \), and let \( A(x, b) \) denote the complex distance from \( x_0 \) to \( \xi(x, b) \).

**Theorem 2.2.** For \( f \in C_0^\infty(X) \) define the Fourier transform \( \hat{f} \) by

\[
\hat{f}(\lambda, b) = \int_X f(x) \exp[-i\lambda + \rho(A(x, b))]dx \quad (\lambda \in \mathfrak{A}^\ast, b \in B).
\]

Then
\[
\int_X |f(x)|^2 \, dx = \int_{\mathbb{R}^* \times B} |f(\lambda, b)|^2 \, c(\lambda) \, d\lambda \, db,
\]
where \( db \) is a suitably normalized \( K \)-invariant measure on \( B \).

**Remarks.** (i) In view of the analogy between horocycles in \( X \) and hyperplanes in \( \mathbb{R}^n \), formula (4) corresponds exactly to the Fourier inversion formula in \( \mathbb{R}^n \) when written in polar coordinate form.

(ii) If \( f \) is a \( K \)-invariant function on \( X \), Theorem 2.2 reduces to Harish-Chandra’s Plancherel formula [6, p. 612]. Nevertheless, Theorem 2.2 can be derived from Harish-Chandra’s formula.

(iii) A “plane wave” on \( X \) is by definition a function on \( X \) which is constant on each member of a family of parallel horocycles. Writing (4) in the form

\[
\int_B f_b(x) \, db
\]

we get a continuous decomposition of \( f \) into plane waves. On the other hand, if we write (4) in the form

\[
f(x) = \int_{\mathbb{R}^*} f_\lambda(x) \, c(\lambda) \, d\lambda
\]

we obtain a decomposition of \( f \) into simultaneous eigenfunctions of all \( D \in D(X) \).

We now define for \( \mathcal{E} \) the analogs of the spherical functions on \( X \).

**Definition.** A distribution (resp. \( C^\infty \) function) on \( \mathcal{E} = G/MN \) is called **conical** if it is (1) \( MN \)-invariant; (2) eigendistribution (resp. eigenfunction) of each \( \mathcal{D} \).

Let \( \xi_0 = MN \), \( \xi^* = m^*MN \), where \( m^* \) is any element in \( M' \) such that the automorphism \( a \to m^*am^{*-1} \) of \( A \) maps \( \rho \) into \( -\rho \). By the Bruhat lemma, \( \mathcal{E} \) will consist of finitely many \( MNA \)-orbits; exactly one, namely \( \mathcal{E}^* = MNA \cdot \xi^* \), has maximum dimension and given \( \xi \in \mathcal{E}^* \) there exists a unique element \( a(\xi) \in A \) such that \( \xi \in MNa(\xi) \cdot \xi^* \) [10, p. 673]. Using Theorem 1.2 we find:

**Theorem 2.3.** Let \( T \) be a conical distribution on \( \mathcal{E} \). Then there exists \( a\psi \in C^\infty(\mathcal{E}^*) \) such that \( T = \psi \) on \( \mathcal{E}^* \) and a linear function \( \mu \): \( \mathbb{R} \to \mathbb{C} \) such that

\[
\psi(\xi) = \psi(\xi^*) \exp[\mu(\log a(\xi))] \quad (\xi \in \mathcal{E}^*).
\]
In general $\psi$ is singular on the lower-dimensional $MNA$-orbits. However, we have:

**Theorem 2.4.** Let $\mu : A \to C$ be a linear function and let $\psi \in C^\infty(\mathcal{S}^*)$ be defined by (5). Then $\psi$ is locally integrable on $\mathcal{S}$ if and only if

\begin{equation}
\text{Re} \langle \alpha, \mu + \rho \rangle > 0 \quad (\text{Re} = \text{real part})
\end{equation}

for each restricted root $\alpha > 0$; here $\langle , \rangle$ denotes the inner product on $A^*$ induced by the Killing form of $\mathfrak{g}$. If (6) is satisfied then $\psi$, as a distribution on $\mathcal{S}$, is a conical distribution.

**Theorem 2.5.** The conical functions on $\mathcal{S}$ are precisely the functions $\psi$ given by (5) where for each restricted root $\alpha > 0$,

\begin{equation}
\frac{\langle \mu, \alpha \rangle}{\langle \alpha, \alpha \rangle} \text{ is an integer } \geq 0.
\end{equation}

**Definition.** A representation $\pi$ of $G$ on a vector space $E$ will be called (1) spherical if there exists a nonzero vector in $E$ fixed by $\pi(K)$; (2) conical if there exists a nonzero vector in $E$ fixed by $\pi(MN)$.

The correspondence between spherical functions on $X$ and spherical representations is well known. In order to describe the analogous situation for $\mathcal{S}$, for an arbitrary function $\phi$ on $\mathcal{S}$, let $E_\phi$ denote the vector space spanned by the $G$-translates of $\phi$ and let $\pi_\phi$ denote the natural representation of $G$ on $E_\phi$.

**Theorem 2.6.** The mapping $\psi \mapsto \pi_\psi$ maps the set of conical functions on $\mathcal{S}$ onto the set of finite-dimensional, irreducible conical representations of $G$. The mapping is one-to-one if we identify proportional conical functions and identify equivalent representations. Also

\[ \psi(g \cdot \xi_0) = (\pi_\psi(g^{-1})e, e'), \]

where $e$ and $e'$, respectively, are contained in the highest weight spaces of $\pi_\psi$ and of its contragredient representation. Finally, $\mu$ in (5) is the highest weight of $\pi_\psi$.

**Corollary 2.7.** Let $\pi$ be a finite-dimensional irreducible representation of $G$. Then $\pi$ is spherical if and only if it is conical.

The highest weights of these representations are therefore characterized by (7). Compare Sugiura [13], where the highest weights of the spherical representations are determined.

3. **The case of a complex $G$.** If $G$ is complex, $M$ is a torus and some of the results of §2 can be improved. Let $\mathfrak{H}$ be a Cartan subalgebra
of $\mathfrak{G}$ containing $\mathfrak{A}$ and $H$ the corresponding analytic subgroup of $G$. Now we assume $G$ simply connected.

Let $D(G/N)$ denote the algebra of all $G$-invariant differential operators on $G/N$. Let $v_0, v^* \in G/N$ be constructed similarly as $\xi_N$ and $\xi^*$ in §2. Then §1 applies to the submanifold $W = H \cdot v^*$ of $V = NH \cdot v^*$ and for each differential operator $D$ on $G/N$, $\Delta(D)$ is defined and can be viewed as a differential operator on $H$.

**Theorem 3.1.** The mapping $\Delta(D)$ is an isomorphism of $D(G/N)$ onto the (real) symmetric algebra $S(\mathfrak{H})$. In particular, $D(G/N)$ is commutative.

As a consequence one finds that the $N$-invariant eigenfunctions $f \in C^\infty(G/N)$ of all $D \in D(G/N)$ have a representation analogous to (5) in terms of the characters of $H$. Let $E_f$ denote the vector space spanned by the $G$-translates of $f$ and let $\pi_f$ be the natural representation of $G$ on $E_f$.

**Theorem 3.2.** The mapping $\pi_f$ is a one-to-one mapping of the set of $N$-invariant holomorphic eigenfunctions of all $D \in D(G/N)$ (proportional $f$ identified) onto the set of all finite-dimensional irreducible holomorphic representations of $G$ (equivalent representations identified). Moreover

$$f(g \cdot v_0) = (\pi_f(g^{-1})e, e'),$$

where $e$ and $e'$, respectively, are contained in the highest weight spaces of $\pi_f$ and of its contragredient representation.

**References**


* Compare the problem indicated in [3, p. 553].

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