It has been shown by Birkhoff [2], [3] that Hilbert's projective metric [4] may be applied to a variety of problems involving linear mappings of a function space into itself. In this note we shall point out that essentially the same metric may be applied to some nonlinear mappings which frequently arise in dynamic programming [1].

Let \( X \) be some set, and let \( P \) denote the set of all nonnegative real-valued functions which have domain \( X \) and are not identically zero. We define an extended real-valued function \( \theta \) on \( P \times P \) as follows:

\[
\theta(f, g) = \log \left( \frac{\sup_{x \in X} f(x)}{\sup_{x \in X} g(x)} \cdot \frac{\sup_{x \in X} g(x)}{\sup_{x \in X} f(x)} \right).
\]

In computing the ratios, we take \( 0 \div 0 \) to be 1, and \( a \div 0 \) to be \( +\infty \) if \( a \neq 0 \). It is easy to show that \( \theta \) is an extended pseudo-metric on \( P \). \( \theta(f, g) = 0 \) implies that \( f = \lambda g \) for some constant \( \lambda > 0 \). We say that a subset \( P^* \) of \( P \) is "metric" if \( \theta \) is an extended metric on \( P^* \). That is, if for any \( f, g \in P^* \), \( \theta(f, g) = 0 \) if and only if \( f = g \).

Let \( L \) be a map of \( P \) into \( P \). If

\[
\sup_{x \in X} \frac{L_f(x)}{L_g(x)} < \sup_{x \in X} \frac{f(x)}{g(x)}
\]

for all \( f, g \in P \), such that \( 0 < \theta(f, g) < \infty \) then we say \( L \) is "ratio reducing on \( P \)." Note that if \( L \) is ratio reducing on \( P \) it follows at once that \( \theta(Lf, Lg) < \theta(f, g) \) for all \( f, g \in P \) such that \( 0 < \theta(f, g) < \infty \).

Thus \( L \) is a contraction mapping with respect to the pseudo-metric \( \theta \). Similar definitions apply on any subset of \( P \). Many linear transformations have been shown [2], [3] to be ratio reducing (or at least ratio nonincreasing). A family \( \{L_{\lambda}\} \) (\( \lambda \) ranging over some set of parameters \( \Lambda \)) is said to be "uniformly ratio reducing" if, given \( f, g \),

\[
\sup_{x \in X} \frac{L_{\lambda}(f(x))}{L_{\lambda}(g(x))} \leq \sup_{x \in X} \frac{f(x)}{g(x)} - \delta_{f, g} \quad \text{for all } \lambda \in \Lambda,
\]

where \( \delta_{f, g} > 0 \) may depend on \( f \) and \( g \) but does not depend on \( \lambda \). Note that if \( \Lambda \) is a finite set then the family \( \{L_{\lambda}\} \) is uniformly ratio reducing if each of its members is ratio reducing.

**Theorem.** If the family \( \{L_{\lambda}: \lambda \in \Lambda\} \) is uniformly ratio reducing,
then the transformation \( L^1 \) defined by
\[
L^1(f(x)) = \sup_{\lambda \in \Delta} L_\lambda(f(x))
\]
is ratio reducing. If in addition \( L_\lambda(g(x)) > \delta_0 > 0 \) for each \( g \in P \) and all \( \lambda \in \Delta \), then the transformation \( L^2 \) defined by
\[
L^2(f(x)) = \inf_{\lambda \in \Delta} L_\lambda(f(x))
\]
is also ratio reducing.

The proof of the theorem is by straightforward computation. To illustrate the application of this theorem to dynamic programming, let us consider a class of problems referred to as “equations of type III” [1, pp. 125–129]. Suppose we are confronted with a system which may be in any one of \( N+1 \) states (call the states \( s_0, s_1, \ldots, s_N \)), and we are trying to drive the system into state \( s_0 \). At each stage, we begin by knowing a probability distribution \( p = (p_0, p_1, \ldots, p_N) \), where \( p_i \) = probability that the system is in state \( s_i \). We may either observe the system (at a cost \( b > 0 \)), or we may perform an operation \( T_i \) on it which will alter the probability distribution in some way at a cost \( a_i > 0 \) \( (i = 1, 2, \ldots, n) \). Then if \( f(p) \) represents the expected cost of driving the system into state \( s_0 \) given that it is initially “known” to be in state \( s_i \) with probability \( p_i \), we see that \( f \) must obey the functional equation

\[
f(p) = \inf \left\{ \sum_{i=1}^{N} p_i f(s_i) + b, f(T_i p) + a_i \right\}
\]

where \( s_i \) denotes the probability distribution which assigns probability 1 to state \( s_i \).

**Theorem.** There is at most one bounded positive solution to the equation (*).

**Proof.** Let \( X \) be the set of all possible distributions over the \( N+1 \) possible states with the exception of \( (1, 0, \ldots, 0) \). This point \( (s_0) \) is in the closure of \( X \). Since the final operation on the system must be an observation, we see that \( f(p) \geq b \). If \( f \) is bounded, it immediately follows that \( \lim_{p \to s_0} f(p) = b \). Let us restrict our attention to the metric subset \( P^* \) of \( P \) consisting of bounded \( f \) such that \( \lim_{p \to s_0} f(p) = b \).

\[
L_0(f(p)) = \sum_{i=1}^{n} p_i f(s_i) + b,
\]

\[
L_i(f(p)) = f(T_i p) + a_i, \quad i = 1, 2, \ldots, n,
\]
are all ratio-reducing on $P^*$. Thus by our Theorem above
\[ L(f(p)) = \inf_{f=0,1,\ldots,n} L_t(f(p)) \]
is ratio-reducing on $P^*$. Hence, if $f$ and $g$ are distinct elements of $P^*$, then $\theta(Lf, Lg) < \theta(f, g)$, which proves there can be at most one bounded solution to $f = Lf$.

A similar method may be applied when the system may be in any one of a continuum of states. Note that in addition to proving the uniqueness of the solution (if any) to (*), the above argument shows that if $g \subseteq P^*$, and \{\$g^n\$\} contains a uniformly convergent subsequence, then \{\$g^n\$\} converges uniformly to the solution of (*).

**BIBLIOGRAPHY**


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