NONLINEAR MONOTONE OPERATORS AND CONVEX SETS IN BANACH SPACES

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Introduction. Let $X$ be a real Banach space, $X^*$ its conjugate space, $(w, u)$ the pairing between $w$ in $X^*$ and $u$ in $X$. If $C$ is a closed convex subset of $X$, a mapping $T$ of $C$ into $X^*$ is said to be monotone if

$$(Tu - T_v, u - v) \geq 0$$

for all $u$ and $v$ in $C$.

It is the object of the present note to prove the following theorem:

**Theorem 1.** Let $C$ be a closed convex subset of the reflexive Banach space $X$ with $0 \in C$, $T$ a monotone mapping of $C$ into $X^*$. Suppose that $T$ is continuous from line segments in $C$ to the weak topology of $X^*$ while $(Tu, u)/\|u\| \to +\infty$ as $\|u\| \to +\infty$.

Then for each given element $w_0$ of $X^*$, there exists $u_0$ in $C$ such that

$$(Tu_0 - w_0, u_0 - v) \leq 0$$

for all $v$ in $C$.

If $C = X$, Theorem 1 asserts that $Tu_0 = w_0$ and reduces to a theorem on monotone operators proved independently by the writer [1] and G. J. Minty [9] and applied to nonlinear elliptic boundary value problems by the writer in [2], [3], and [6]. (See also Leray and Lions [7].) If $C = V$, a closed subspace of $X$, the conclusion of Theorem 1 is that $Tu_0 - w_0 \in V^\perp$, which yields a variant of the generalized form of the Beurling-Livingston theorem proved by the writer in [4] and [5]. The conclusion of Theorem 1 for $C = X$ was extended by the writer to classes of densely defined operators (see [6] for references) and in [5] to multivalued mappings.

It is easily shown that Theorem 1 generalizes and includes as a special case the following linear theorem of Stampacchia, which has been applied by the latter to the proof of the existence of capacitary potentials with respect to second-order linear elliptic equations with discontinuous coefficients:

**Theorem 2.** Let $H$ be a real Hilbert space, $C$ a closed convex subset of $H$, $a(u, v)$ a bilinear form on $H$ which is separately continuous in $u$.
and \( v \). Suppose that there exists a constant \( c > 0 \) such that \( a(u, u) \geq c\|u\|^2 \) for all \( u \) in \( H \).

Then for each \( w_0 \) in \( H \), there exists \( u_0 \) in \( C \) such that

\[
a(u_0, u_0 - v) \leq (w_0, u - v)
\]

for all \( v \) in \( C \).

1. We denote weak convergence by \( \rightarrow \), strong convergence by \( \rightarrow \).

**Lemma 1.** If \( u_0 \in C \), \( u_0 \) is a solution of the inequality (2) if and only if

\[
(Tv - w_0, v - u_0) \geq 0
\]

for all \( v \) in \( C \).

**Proof of Lemma 1.** If for a given \( u_0 \) in \( C \) and all \( v \) in \( C \), we have

\[
(Tu_0 - w_0, u_0 - v) \leq 0,
\]

then since

\[
(Tu_0 - Tv, u_0 - v) \geq 0
\]

by monotonicity, it follows that

\[
(Tv, u_0 - v) \leq (Tu_0, u_0 - v) \leq (w_0, u_0 - v),
\]

i.e.,

\[
(Tv - w_0, v - u_0) \leq 0.
\]

Conversely, suppose the inequality (4) holds for all \( v \) in \( C \). Suppose \( v_0 \in C \), and for \( 0 < t \leq 1 \), set

\[
v_t = (1 - t)u_0 + tv_0.
\]

Then \( v_t \in C \), \( v_t - u_0 = tv_0 - u_0 \), and we have

\[
0 \leq (Tv_t - w_0, t(v_0 - u_0)) = t(Tv_t - w_0, v_0 - u_0).
\]

Since \( t > 0 \) may be canceled, we have

\[
(Tv_t - w_0, v_0 - u_0) \geq 0.
\]

If we let \( t \to 0 \) and use the weak continuity of \( T \) on segments in \( C \), we have \( Tv_t \to Tu_0 \), and hence

\[
(Tu_0 - w_0, u_0 - v_0) \leq 0. \quad q.e.d.
\]

**Definition.** Let \( c(r) = \inf_{\|u\|=r} \{ (Tu, u)/\|u\| \} \). By the hypothesis of Theorem 1, \( c(r) \to + \infty \) as \( r \to + \infty \). We have

\[
(Tu, u) \geq c(\|u\|)\|u\|, \quad u \in C.
\]

**Lemma 2.** There exists a constant \( M \) which depends only upon the
function $c(r)$ and on $\|w_0\|$ such that if $u_0$ is a solution of the inequality (2), then $\|u_0\| \leq M$.

**Proof of Lemma 2.** If

$$(Tu_0 - w_0, u_0 - v) \leq 0, \quad v \in C,$$

we have since $0 \in C$,

$$c(\|u_0\|) \leq (Tu_0, u_0) \leq (Tu_0 - w_0, u_0) + (w_0, u_0) \leq \|w_0\| \cdot \|u_0\|.$$

Hence

$$c(\|u_0\|) \leq \|w_0\|$$

and

$$\|u_0\| \leq M(\|w_0\|, c(r)). \text{ q.e.d.}$$

**Definition.** If $G \subseteq X \times X^*$, $G$ is said to be a monotone set if $[u, w], [u_1, w_1] \in G$ implies that $(w - w_1, u - u_1) \geq 0$.

$G$ is said to be maximal monotone if it is monotone and maximal in the monotone sets ordered by inclusion.

**Lemma 3.** Under the hypotheses of Theorem 1, suppose that $C$ has $0$ as an interior point and let $G \subseteq X \times X^*$ be given by

$$G = \{[u, w] | u \in C, w = Tu + z, \text{ where } (z, u - v) \geq 0 \text{ for all } v \text{ in } C\}.$$

Then $G$ is a maximal monotone set in $X \times X^*$.

**Proof of Lemma 3.** $G$ is a monotone set since if $[u, w]$ and $[u_1, w_1] \in G$, with $w = Tu + z, w_1 = Tu_1 + z_1$, then

$$(w - w_1, u - u_1) = (Tu - Tu_1, u - u_1) + (z, u - u_1) + (z_1, u_1 - u) \geq 0.$$

Suppose on the other hand that $[u_0, w_0] \in X \times X^*$ with

$$(w_0 - w, u_0 - u) \geq 0$$

for all $[u, w]$ in $G$. We assert first that $u_0 \in C$. Otherwise, $u_0 = sv_0$ for some $v_0$ on the boundary of $C$ with $s > 1$. Let $z_0 = 0$ be an element of $X^*$ such that $(z_0, v_0 - v) \geq 0$ for all $v$ in $C$. Since $0$ is an interior point of $C$, $(z_0, v_0) > 0$. For each $\lambda > 0$, $[v_0, T\lambda v_0 + \lambda z_0]$ lies in $G$. Hence

$$0 \leq (w_0 - T\lambda v_0 - \lambda z_0, u_0 - v_0) = (s - 1)(w_0 - T\lambda v_0 - \lambda z_0, v_0).$$

Cancelling $(s - 1) > 0$, we have

$$\lambda(z_0, v_0) \leq (w_0, v_0) - (T\lambda v_0, v_0).$$
which is a contradiction since \((z_0, v_0) > 0\) and \(\lambda\) is arbitrary. Hence \(u_0 \in C\).

In addition, for each \(u\) in \(C\), \([u, Tu]\) lies in \(G\). Hence
\[
(Tu - w_0, u - u_0) \geq 0.
\]
Applying Lemma 1, we have
\[
(Tu_0 - w_0, u_0 - v) \leq 0, \quad v \in C.
\]
Hence \(Tu_0 - w_0 = -z\), where \((z, u_0 - v) \geq 0\) for all \(v\) in \(C\). Hence \(w_0 = Tu_0 + z\), and \([u_0, w_0] \in G\). q.e.d.

**Lemma 4.** Theorem 1 holds if \(X\) is a finite dimensional Banach space \(F\).

**Proof of Lemma 4.** We may suppose without loss of generality that \(w_0 = 0\), that \(F\) is a finite dimensional Hilbert space with \(F^* = F\), and that \(C\) spans \(F\) and hence has an interior point \(v_0\) in \(F\). Replacing \(C\) by \(C_0 = v_0 - C\) and defining a new mapping \(T'\) on \(C_0\) by \(T'u = -T(v_0 - u)\), it is easy to verify that we may assume that \(0\) is an interior point of \(C\) and the condition on \((Tu, u)\) is replaced by
\[
(Tu, u - v) \geq c(||u||)||u||
\]
for a given \(v_0\) in \(C\), with \(c(r) \to +\infty\) as \(r \to +\infty\).

Let \(G\) be the maximal monotone set in \(FXF^*\) constructed in Lemma 3. Then \(nG\) is maximal monotone for each positive integer \(n\). By a theorem of Minty [8], for each \(n > 0\), there exists \([u_n, w_n] \in G\) such that
\[
u_n + nw_n = 0.
\]
Since \(w_n = Tu_n + z_n\), where \((z_n, u_n - v) \geq 0\) for all \(v\) in \(C\), we have
\[
-\left(\frac{1}{n} u_n, u_n - v_0\right) = (w_n, u_n - v_0) = (Tu_n, u_n - v_0) + (z_n, u_n - v_0) \geq c(||u_n||)||u_n||,
\]
while
\[
-\left(\frac{1}{n} u_n, u_n - v_0\right) \leq \frac{1}{n} ||u_n|| \cdot ||v_0||.
\]
Thus \(c(||u_n||) \leq n^{-1}||v_0||\), and \(||u_n|| \leq M\), independent of \(n\). We may extract a subsequence which we again denote by \(u_n\) such that \(u_n \to u_0\) in \(F\). Then \(w_n \to 0\). For each \(u\) in \(C\)
\[
(Tu - w_n, u = u_n) \geq 0.
\]
Taking the limit as $n \to \infty$, we have

$$(Tu, u - u_0) \geq 0, \quad u \in C.$$ 

By Lemma 1,

$$(Tu_0, u_0 - v) \leq 0$$

for all $v$ in $C$. q.e.d.

**Proof of Theorem 1.** It suffices to take $w_0 = 0$. For each finite dimensional subspace $F$ of $X$, let $C_F = C \cap F$, $j_F$ be the injection map of $F$ into $X$, $j_F^*$ the dual projection map of $X^*$ onto $F^*$. We set

$$T_F = j_F^*(T|C_F): C_F \to F^*.$$ 

Then $T_F$ satisfies the hypotheses of Lemma 4, and there exists $u_F$ in $C_F$ such that

$$(T_Fu_F, u_F - v) = (Tu_F, u_F - v) \leq 0, \quad v \in C_F.$$ 

By Lemma 2, since for $u$ in $C_F$,

$$(T_Fu, u) = (Tu, u) \geq c(u, u),$$

there exists a constant $M$ independent of $F$ such that $||u_F|| \leq M$. Since $X$ is reflexive and $C$ is weakly closed, there exists $u_0$ in $C$ such that for every finite dimensional $F$, $u_0$ lies in the weak closure of the set $V_F = \bigcup_{F \subseteq F_1} \{u_F\}$.

Let $v$ be an arbitrary element of $C$, $F$ a finite dimensional subspace of $X$ which contains $v$. For $u_{F_1}$ in $V_F$, by Lemma 1,

$$(Tv, v - u_{F_1}) \geq 0.$$ 

Since $(Tv, v - v_1)$ is weakly continuous in $v_1$, we have

$$(Tv, v - u_0) \leq 0, \quad v \in C.$$ 

By Lemma 1, $(Tu_0, u_0 - v) \geq 0$ for $v$ in $C$. q.e.d.

**Bibliography**


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