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COMPLETELY 0-SIMPLE AND HOMOGENEOUS
n REGULAR SEMIGROUPS

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1. In this note we state three new results (Theorems 1, 4 and 5) about the completely 0-simple and homogeneous \( n \) regular semigroups.

We follow the notation and terminology of [1] unless stated otherwise. Throughout, \( S \) denotes a semigroup with zero. Let \( a \in S \setminus 0 \). Denote by \( V(a) \) the set of all inverses of \( a \) in \( S \), that is, \( V(a) = \{ x \in S : axa = a, xax = x \} \). A semigroup \( S \) with zero is said to be homogeneous \( n \) regular if the cardinal number of the set \( V(a) \) of all inverses of \( a \) is \( n \) for every nonzero element \( a \) in \( S \), where \( n \) is a fixed positive integer. Let \( T \) be a subset of \( S \). We denote by \( E(T) \) the set of all idempotents of \( S \) in \( T \).

2. The next theorem is a generalization of R. McFadden and Hans Schneider's theorem [3].

**Theorem 1.** Let \( S \) be a 0-simple semigroup and let \( n \) be a fixed positive integer. Then the following are equivalent.

(i) \( S \) is a homogeneous \( n \) regular and completely 0-simple semigroup.

(ii) For every \( a \neq 0 \) in \( S \) there exist precisely \( n \) distinct nonzero elements \((x_i)_{i=1}^n\) such that \( axa = a \) for \( i = 1, 2, \ldots, n \) and for all \( c, d \) in \( S \)

\[ cdc = c \neq 0 \text{ implies } dcd = d. \]

(iii) For every \( a \neq 0 \) in \( S \) there exist precisely \( n \) distinct nonzero idempotents \((e_i)_{i=1}^k = E_a \) and \( k \) distinct nonzero idempotents \((f_i)_{i=1}^k = F_a \) such that \( e_i a = a = a f_i \); for \( i = 1, 2, \ldots, h, j = 1, 2, \ldots, k, \) \( h k = n, E_a \text{ contains every nonzero idempotent which is a left unit of } a, F_a \text{ contains every nonzero idempotent which is a right unit of } a \) and \( E_a \cap F_a \text{ contains at most one element} \).

(iv) For every \( a \neq 0 \) in \( S \) there exist precisely \( k \) nonzero principal right ideals \((R_i)_{i=1}^k\) and \( h \) nonzero principal left ideals \((L_i)_{i=1}^k\) such that

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$R_i$ and $L_j$ contain $h$ and $k$ inverses of $a$, respectively, every inverse of $a$ is contained in a suitable set $R_i \cap L_j$ for $i = 1, \ldots, k$, $j = 1, \ldots, h$, and $R_i \cap L_j$ contains at most one nonzero idempotent, where $hk = n$.

(v) Every nonzero principal right ideal $R$ contains precisely $h$ nonzero idempotents and every nonzero principal left ideal $L$ contains precisely $k$ nonzero idempotents such that $hk = n$, and $R \cap L$ contains at most one nonzero idempotent.

(vi) $S$ is completely 0-simple. For every 0-minimal right ideal $R$ there exist precisely $h$ 0-minimal left ideals $(L_i)_{i=1}^h$ and for every 0-minimal left ideal $L$ there exist precisely $k$ 0-minimal right ideals $(R_j)_{j=1}^k$ such that $LR_i = L_iR = S$, for every $i = 1, \ldots, h$, $j = 1, \ldots, k$, where $hk = n$.

(vii) $S$ is completely 0-simple. Every 0-minimal right ideal $R$ of $S$ is the union of a right group with zero $G^0$, a union of $h$ disjoint groups except zero, and a zero subsemigroup $Z$ which annihilates the right ideal $R$ on the left and every 0-minimal left ideal $L$ of $S$ is the union of a left group with zero $G^0$, a union of $k$ disjoint groups except zero, and a zero subsemigroup $Z'$ which annihilates the left ideal $L$ on the right and $hk = n$.

(viii) $S$ contains at least $n$ nonzero distinct idempotents, and for every nonzero idempotent $e$ there exists a set $E$ of $n$ distinct nonzero idempotents of $S$ such that $eE$ is a right zero subsemigroup of $S$ containing precisely $h$ nonzero idempotents, $Ee$ is a left zero subsemigroup of $S$ containing precisely $k$ nonzero idempotents of $S$, $e(E(S) \setminus E) = (0) = (E(S) \setminus E)e$, and $eE \cap Ee = \{e\}$, where $hk = n$.

If $n = 1$, then the theorem above takes the same form as R. McFadden and Hans Schneider's theorem [3], except (iv).

3. The following lemmas and Theorem 2 contain main ideas to prove the theorem above.

**Lemma 1.** For all $a, b$ in a Rees matrix semigroup $S = M^0(G; I, \Lambda; P)$, $aba = a \neq 0$ implies $bab = b$. Every completely 0-simple semigroup has this property by Theorem 3.5 [1].

**Lemma 2.** In a completely 0-simple semigroup $S$, for a nonzero idempotent $e$ and a nonzero element $a$ in $S$ such that $ea = a$ ($ae = a$) the equation $ax = e$ ($xa = e$) has a solution $x$ in $Se(eS)$. If we denote by $x_0$ a solution of the equation above, then $x_0$ is an inverse of $a$, that is, $ax_0a = a$ and $x_0ax_0 = x_0$.

**Lemma 3.** Let $S$ be a completely 0-simple semigroup. Let $a \in S \setminus \{0\}$. If $E_a = (e_i)_{i=1}^h$ and $F_a = (f_j)_{j=1}^k$ are sets of all nonzero idempotents of $S$ such that $e_i a = a = af_j$ for every $i = 1, 2, \ldots, h$, $j = 1, 2, \ldots, k$, $|E_a|$
**Theorem 2.** A nonzero element \( a \) in a completely 0-simple semigroup \( S \) has precisely \( n \) inverses if and only if the 0-minimal right and left ideals of \( S \) containing \( a \) contain respectively \( h \) and \( k \) nonzero idempotents of \( S \) such that \( hk = n \).

**Remark.** Theorem 2 is a corollary of the following theorem.

**Theorem.** A nonzero element \( a = (g)_{ij} \) in a Rees matrix semigroup \( S = M^0(G; I, \Lambda; P) \) has precisely \( h \) inverses if and only if \( R_i = ((a)_{ij}; a \in G, j \in \Lambda) \) and \( L_j = ((a)_{ij}; a \in G, i \in I) \) contain precisely \( h \) and \( k \) nonzero idempotents of \( S \), respectively, with \( hk = n \), where \( i \in I, j \in \Lambda, 0 \neq g \in G \).

Notice that there is no condition of regularity in the theorem.

4. \( S \) is said to be \( h-k \) type if every nonzero principal left ideal of \( S \) contains precisely \( k \) nonzero idempotents and every nonzero principal right ideal of \( S \) contains precisely \( h \) nonzero idempotents of \( S \). A regular semigroup \( S \) is said to be \( h-k \) regular if \( S \) is \( h-k \) type. A generalization of P. S. Venkatesan's theorem \([5]\) follows.

**Theorem 3.** (1) A regular semigroup \( S \) with zero is 1-n type in which every nonzero idempotent is primitive if and only if \( S \) is the union of its 0-minimal ideals each of which is a 1-n type homogeneous \( n \) regular and completely 0-simple semigroup.

(2) The following statements on a semigroup \( S \) with zero are equivalent.

(i) \( S \) is regular and for any nonzero idempotent \( e \) in \( S \) the equation \( exe = e \) has precisely \( n \) distinct idempotent solutions \( U(e) = (e_i; i = 1, 2, \ldots, n) \) including \( e \) such that \( e \) is a right unit of \( U(e) \) and \( e \) is the left zero of \( U(e) \).

(ii) Every nonzero principal right ideal of \( S \) is 0-minimal and is generated by just one idempotent. Every nonzero principal left ideal of \( S \) is 0-minimal and is generated by a nonzero idempotent containing precisely \( n \) distinct nonzero idempotents.

(iii) For each nonzero \( a \) in \( S \) there exists a unique set \( U(a) = (a_i; aa = a, i = 1, \ldots, n) \) such that there exist a nonzero principal left ideal containing \( U(a) \) and \( n \) distinct nonzero principal right ideals each of which contains just one element of \( U(a) \). Every set \( (Sb \cap cS) \) contains at most one nonzero idempotent, for \( b, c \) in \( S \).

(iv) For every nonzero element \( a \) in \( S \) there exist a unique idempotent \( e \) and a set \( (f_i; i = 1, 2, \ldots, n) \) of nonzero idempotents such that
ea = a = af_i (i = 1, 2, · · · , n). Every nonzero principal right ideal contains just one nonzero idempotent and every nonzero principal left ideal contains precisely n nonzero idempotents.

(v) S is a 1—n type regular semigroup and if f is a nonzero idempotent such that f ∈ E(Se \ 0) then fE(Se) = E(Se)f = (0).

(vi) S is a 1—n type regular semigroup and for any a, b and c in S \ 0, 0 ≠ ab = cb implies a = c.

5. W. D. Munn defined the Brandt congruence [4]. A congruence ρ on a semigroup S with zero is called a Brandt congruence if S/ρ is a Brandt semigroup. If S is a 1 — n (or n — 1) type homogeneous n regular and completely 0-simple semigroup, then there is a Brandt congruence.

THEOREM 4. Let S be a 1—n type homogeneous n regular and completely 0-simple semigroup. Define a relation ρ on S in such a way that a ρ b if and only if there exists a set (ei)_{i=1}^{n} of n distinct nonzero idempotents such that e_i a = e_i b ≠ 0, for every i = 1, 2, · · · , n. Then ρ is an equivalence on S \ 0. If we extend ρ on S by defining (0) to be a ρ-class on S, then ρ is a proper Brandt congruence on S. Furthermore, if σ is any proper Brandt congruence on S, then ρ ⊂ σ.

THEOREM 5. Let S be a 1—n regular semigroup in which every nonzero idempotent is primitive. If we define a relation ρ on S by the rule that a ρ b if and only if there exists a set (ei)_{i=1}^{n} of n nonzero idempotents in S such that e_i a = e_i b ≠ 0 (i = 1, 2, · · · , n). Then ρ is an equivalence on S \ 0. If we extend ρ to S by defining (0) to be a ρ-class on S, then ρ is a proper congruence on S such that S/ρ is an inverse semigroup. Furthermore, ρ is the finest such congruence.

We list more theorems.

THEOREM 6. If S is a h—k type semigroup with zero and if every nonzero idempotent of S is primitive, then SeS (e ∈ E(S \ 0)) is a completely 0-simple and h — k type homogeneous hk regular semigroup.

THEOREM 7. Let n, h_n and k_n be positive integers with h_n k_n = n. A regular semigroup S with zero is h_n—k_n type in which every nonzero idempotent is primitive if and only if S is a union of its minimal ideals, each of which is a h_n—k_n type homogeneous n regular and completely 0-simple semigroup.

THEOREM 8. The following statements on a semigroup S with zero are equivalent (see [2, Theorem 3] and [5]).

(i) S is h_n—k_n regular. For all a, x in S axa = a ≠ 0 implies xax = x.
(ii) $S$ is $h_n - k_n$ regular. For $a$, $b$, $x$ and $y$ in $S$ $xa = sb 
eq 0$ and $ay = by 
eq 0$ implies $a = b$.

(iii) $S$ is $h_n - k_n$ regular. For every $e$ in $E(S\setminus 0)$ there exists a set $I$ of $n$ nonzero idempotents such that $eI$ and $Ie$ are right and left zero subsemigroups of $S$, respectively, $eI \cap Ie = (e)$ and $e(E(S\setminus I)) = (0) = E(S\setminus I)e$.

(iv) Every nonzero principal right (left) ideal of $S$ is 0-minimal and is generated by a nonzero idempotent containing precisely $h_n$ ($k_n$) nonzero idempotents. $(a \cup aS) \cap (b \cup Sb)$ contains at most one nonzero idempotent, for $a$, $b$ in $S$, where $h_n k_n = n$.

REFERENCES


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