A JORDAN DECOMPOSITION FOR OPERATORS IN BANACH SPACE

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Operators $T$ with real spectrum in finite dimensional complex Euclidian space may be characterized by the property

$$e^{itT} = O(t^k), \quad t \text{ real.}$$

Our result is a Jordan decomposition theorem for operators $T$ in reflexive Banach space which satisfy (1) and whose spectrum (which is real because of (1)) has linear Lebesgue measure zero.

1. The Jordan manifold. Let $X$ be a complex Banach space; denote by $B(X)$ the Banach algebra of all bounded linear operators acting on $X$. For $m = 0, 1, 2, \cdots$, $C^m$ is the topological algebra of all complex valued functions on the real line $\mathbb{R}$ with continuous derivatives up to the order $m$, with pointwise operations and with the topology of uniform convergence on every compact set of all such derivatives. Fix $T \in B(X)$. Following [3], we say that $T$ is of class $C^m$ if there exists a $C^m$-operational calculus for $T$, i.e., a continuous representation $f \mapsto T(f)$ of $C^m$ into $B(X)$ such that $T(1) = I$, $T(f) = T$ if $f(t) = t$, and $T(\cdot)$ has compact support. The latter is then equal to the spectrum of $T$, $\sigma(T)$. It is known that if $T$ satisfies (1), then it is of class $C^m$ for $m \geq k+2$ and has real spectrum (cf. Lemma 2.11 in [3]).

From now on, let $T \in B(X)$ satisfy (1), and let $T(\cdot)$ be the (unique) $C^m$-operational calculus for $T$, for $m$ fixed $\geq k+2$. We write:

1. $|f|_{m,T} = \sum_{|f|_{m,T} \leq 1} \max_{|f|_{m,T} = 1} |f| f \in C^m$;
2. $\|f\|_{m,T} = \sup \{|T(f)x| : f \in C^m, |f|_{m,T} \leq 1\}, x \in X$;
3. $D_m = \{x \in X : |x|_{m,T} < \infty\}$;
4. $D = \cup_{m \geq k+3} D_m$.

We call $D$ the Jordan manifold for $T$. It is an invariant linear manifold for any $V \in B(X)$ which commutes with $T$. If $\sigma(T)$ is a finite union of points and closed intervals, then there exists an $m \geq k+2$ such that $D = D_m = X$. This is true for $m = k+2$ if $\sigma(T)$ is a finite point set. It follows in particular that $D_{k+2}$ contains every finite dimensional invariant subspace for $T$, hence all the eigenvectors of $T$. It is also true that $D$ contains all the root vectors for $T$, and is therefore dense in $X$ if the root vectors are fundamental in $X$.

Theorem 1. Suppose that all nonzero points of $\sigma(T)$ are isolated.
Then the closure of \( D_{k+2} \) contains the closed range of \( T^{k+1} \). For \( k = 0 \) and \( X \) reflexive, \( D_2 \) is dense in \( X \).

2. The Jordan decomposition. If \( W \) is a linear manifold in \( X \), we denote by \( T(W) \) the algebra of all linear transformations of \( X \) with domain \( W \) and range contained in \( W \).

Let \( B \) denote the Borel field of \( R \).

A generalized spectral measure on \( W \) is a map \( E(\cdot) \) of \( B \) into \( T(W) \) such that

(i) \( E(R)x = x \) for all \( x \in W \), and

(ii) \( E(\cdot)x \) is a bounded regular strongly countably additive vector measure on \( B \), for each \( x \in W \).

We can state now our generalization of the classical Jordan decomposition theorem for complex matrices with real spectrum to infinite dimensional Banach spaces.

**Theorem 2.** Let \( X \) be a reflexive Banach space. Let \( T \in B(X) \) satisfy (1). Suppose \( \sigma(T) \) (which lies on \( R \) because of (1)) has linear Lebesgue measure zero. Let \( D \) be the Jordan manifold for \( T \). Then there exist \( S \) and \( N \) in \( T(D) \) such that

(a) \( T/D = S + N \);
(b) \( SN = NS \);
(c) \( N^{k+1} = 0 \); and
(d) \( p(S)x = \int_{\sigma(T)} p(t) \, dE(t)x, \ x \in D \)

for all polynomials \( p \), where \( E(\cdot) \) is a generalized spectral measure on \( D \) supported by \( \sigma(T) \) and commuting with any \( V \in B(X) \) which commutes with \( T \).

This decomposition is “maximal-unique,” meaning that if \( W \) is an invariant linear manifold for \( T \) for which (a)–(d) are valid with \( W \) replacing \( D \), then \( W \subset D \) and the transformations \( S, N \) and \( E(b) \) \((b \in B)\) corresponding to \( W \) are the restrictions to \( W \) of the respective transformations associated with \( D \).

The proof uses a refinement of the method we applied in the proof of Theorem 3.13 in [3].

It turns out that \( D = D_{k+2} \). For each \( x \in D \), the map \( f \to T(f)x \) of \( C^{k+2} \) into \( X \) has an extension as a continuous linear map of \( C^{k} \) into \( D \) given by

\[
T(f)x = \sum_{|s| \leq k} (1/j!) \int_{\sigma(T)} f^{(s)}(t) \, dE(t)N^s x
\]

(for all \( f \in C^k \) and each \( x \in D \)). The extended map \( f \to T(f) \) of \( C^k \) into \( T(D) \) is multiplicative.
Keeping in mind the usual definition of a resolution of the identity, it is interesting to notice that if \( N \) (or \( S \)) is closable, then \( E(b) \) commutes with \( S \) and \( N \) and \( E(a \cap b) = E(a)E(b) \) for all \( a, b \in B \). This is true in particular if \( k = 0 \), since \( N = 0 \) (cf. (c)) is trivially closable.

Theorem 2 may be given a version fitting into Dunford’s theory of spectral operators [1]. Since \( D = D_{k+2} \), \( D \) is a normed linear space under the norm \( \|x\| = \|x\|_{k+2,T} \). Let us call its completion \( Y \) the Jordan space for \( T \). \( T \) induces in a natural way an operator \( T_Y \in B(Y) \).

**Theorem 2'.** Let \( T \) be as in Theorem 2 (with \( X \) not necessarily reflexive). Then \( (T_Y)^* \) is spectral of class \( Y \) and type \( k \).

The case \( k = 0 \) has a distinguished position if \( X \) is a Hilbert space. By Theorem 5 in [2], Condition (1) by itself is then sufficient for \( T \) to be spectral of scalar type. This is no longer true (in Hilbert space) for \( k \geq 1 \), even when \( \sigma(T) \) is a sequence with 0 as its only limit point. In Banach space (even reflexive) this breaks down even for \( k = 0 \) (cf. [2, p. 176]). Let \( P(\mathbb{R}) \) denote the ring of polynomials over \( \mathbb{R} \). Condition (1) for \( k = 0 \) is equivalent to the condition \( |e^{ip(T)}| < M < \infty \) for all \( p \in P(\mathbb{R}) \) of degree \( \leq 1 \). Dropping this limitation on the degree, we get a criterion for spectrality which is valid in any weakly complete Banach space.

**Theorem 3.** \( T \in B(X) \) is of class \( C \) and has real spectrum if and only if

\[
\sup_{p \in P(\mathbb{R})} |e^{ip(T)}| < \infty.
\]

If \( X \) is weakly complete, Condition (2) is necessary and sufficient for \( T \) to be spectral of scalar type with real spectrum.

The proof uses Theorem 2 in [4].

**References**