RINGS OF MEROMORPHIC FUNCTIONS

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1. Introduction. This paper concerns itself with certain rings of meromorphic functions on noncompact Riemann surfaces. Let Ω denote a noncompact Riemann surface. We denote by A the collection of all mappings of Ω into the complex plane C which are analytic on Ω. Also, we denote by M the collection of all mappings of Ω into the Riemann sphere Σ which are meromorphic on Ω. As is well known, A is an integral domain under the operations of pointwise addition and multiplication, and M is the field of quotients of A. The rings considered here are those subrings of M which contain the ring A. Such subrings will be referred to as *A-rings of M*. In particular, A is itself an A-ring of M, as is the field M.

The ring A has been extensively investigated in recent years, and a considerable amount of information concerning the ideal theory of this ring has been obtained. The main result here is the theorem of Helmer [3], which asserts that every finitely generated ideal of A is actually a principal ideal of A. This theorem is the basis for most of the known results on the ideal theory of A, as is evident from the papers of Henriksen [4], [5], Kakutani [7], and Banaschewski [2].

We announce some results pertaining to the A-rings of M, the principal one of which is a characterization of these rings (Theorem 3). Thanks to this characterization, a number of theorems concerning the ideal theory of A extend to any A-ring of M, as, for example, the theorem of Helmer. Inasmuch as A is itself an A-ring, our results may be considered as generalizations of the corresponding results for A.

The methods involved in the proofs of these results involve a study and exploitation of the valuation theory of M, which was previously considered by Alling [1]. In particular, we make considerable use of the valuation rings of M which are also A-rings of M. These rings are readily identified by means of Helmer's theorem, and they may be employed to prove many of the known results on the ideal theory of A. Moreover, the arguments involved in these proofs frequently apply to any A-ring of M. It is also possible to classify certain A-rings by these methods, and we are able, for example, to determine the noetherian A-rings of M.

Finally, we consider the extent to which a Riemann surface is determined by its A-rings. More exactly, we can show that if two A-

rings of functions mermorphic on two noncompact Riemann surfaces are isomorphic, then the isomorphism in question is induced by a conformal or anticonformal equivalence between the two surfaces. This may be considered as a generalization of a theorem of Nakai [8], who proved this for the case where the A-rings in question are just the rings of functions analytic on the two surfaces. The proof of this result again makes use of the valuation theory of M, especially the characterization of the noetherian valuation rings of M as given by Iss'sa [6]. However, the proof does not depend on the theorem of Nakai, as we derive it from a more general result (Theorem 13) concerning isomorphisms between fields of meromorphic functions. In particular, we obtain the field isomorphism theorem of Iss'sa [6] without the use of the Nakai theorem.

2. **Algebraic preliminaries.** Let D be an integral domain and let K be its field of quotients. We shall say that a nonempty subset S of D is *a multiplicative subset of* D if $0 \notin S$ and if S is closed under multiplication (i.e., if $x \in S$ and $y \in S$, then $xy \in S$). If S is a multiplicative subset of D, the subset $\{x/y: x \in D, y \in S\}$ of the field K, to be denoted by $S^{-1}D$, is a subring of K containing D which will be termed *the ring of quotients of* D *with respect to* S. Further, a subring B of K is called *a ring of quotients of* D if $B = S^{-1}D$ for some multiplicative subset S of D. In the special case $S = D - P$, where $P \neq D$ is a prime ideal of D, the ring of quotients $S^{-1}D$, denoted by D_P, is called *the localization of* D *at* P.

Given a ring of quotients $S^{-1}D$ of D, a number of relations hold between the ideals of $S^{-1}D$ and the ideals of D which do not intersect the set S (cf. [9, pp. 41–49, 218–233]). These relations are employed in the proofs of our results, as are many results from valuation theory.

3. ***A*-rings of *M*.**

DEFINITION 1. *An* A*-ring of* M *is a subring of* M *which contains the ring* A.

Our study of the A-rings of M is based on the following three theorems, and especially on the third.

THEOREM 1. *Let* B *be an* A*-ring of* M *and let* P *be a prime ideal of* B. *Then* B_P, *the localization of* B *at* P, *is a valuation ring of* M *which contains* B. *Conversely, if* R *is a valuation ring of* M *which contains* B, *then* $R = B_P$ *for some prime ideal* P *of* B.

THEOREM 2. *Let* B *be an* A*-ring of* M. *Then* B *is the intersection of a collection of valuation rings of* M.

THEOREM 3. *Let* B *be an* A*-ring of* M. *Then* B *is a ring of quotients of* A. *In fact,* $B = S^{-1}A$, *where* $S = \{f \in A : 1/f \in B\}$.

Thus the A-rings of M are exactly the ring of quotients of A, and the A-rings of M which are also valuation rings of M are exactly the localizations of A at its prime ideals. These results may be used to advantage in studying the A-rings of M. In view of the relations between the ideals of A and those of $S^{-1}A$, we obtain extensions of a number of results on the ideal theory of A to the A-rings of M, such as the following.

THEOREM 4. *Let B be an A-ring of M. Then every finitely generated ideal of B is a principal ideal of B.*

THEOREM 5. *Let B be an A-ring of M and let P be a nonzero, proper prime ideal of B. Then P is contained in exactly one maximal ideal of B.*

THEOREM 6. *Let B be an A-ring of M and let P be a maximal ideal of B. Then the collection of all primary ideals of B which are contained in P is totally ordered under set inclusion.*

THEOREM 7. *Let B be an A-ring of M and let P be a maximal ideal of B. Then the intersection of any collection of prime (resp. primary) ideals of B contained in P is again a prime (resp. primary) ideal of B.*

Since these theorems are all known to be valid for A itself [2], they may be considered as generalizations of the ideal theory of A. One may also obtain a number of results on the valuation rings of M which contain A by the use of our characterization of A-rings. For example, using some results [2], [5] on the prime ideals of A, we have the following.

THEOREM 8. *Let R be a nontrivial valuation ring of M which contains A. Then the following are equivalent: (1) B is a noetherian ring. (2) B is a valuation ring of rank one. (3) B is a valuation ring of finite rank. (4) B is a maximal subring of M. (5) There exists a point $a \in \Omega$ such that $B = \{f \in M : f(a) \neq \infty\}$.*

Of particular interest are those A-rings of M consisting of all functions in M having no poles on a given subset of Ω.

DEFINITION 2. *Given $E \subset \Omega$, we define $A(E) = \{f \in M : f(a) \neq \infty, a \in E\}$.*

Evidently $A(E)$ is the collection of functions $f \in M$ which are analytic at each point of E, so $A(E)$ is an A-ring of M. With suitable restrictions on the set E, the ring $A(E)$ must satisfy some very strong conditions.

THEOREM 9. *Let B be an A-ring of M, $B \neq M$. Then the following are equivalent: (1) $B = A(E)$, where E is a nonempty, relatively compact subset of Ω. (2) B is a noetherian ring. (3) B is a principal ideal ring.*

(4) B is a unique factorization ring. (5) Every proper, nonzero prime ideal of B is a maximal ideal of B. (6) Every proper, nonzero prime ideal of B is a minimal prime ideal of B. (7) Every subring of M which contains B is a noetherian ring. (8) Every valuation ring of M which contains B is a noetherian ring.

The rings described by this theorem can then be used to characterize the rings $A(E)$ with $E \subset \Omega$.

THEOREM 10. *Let B be an A-ring of M. Then $B = A(E)$ for some subset E of Ω if and only if B is the intersection of a decreasing sequence of A-rings which satisfy the conditions of Theorem 9.*

4. **Isomorphism theorems.** In order to determine the possible ring isomorphisms between two A-rings on two noncompact Riemann surfaces, we make use of a recent theorem of Iss'sa [6], which characterizes the noetherian valuation rings of the field M. This result may be stated as follows.

THEOREM 11. *Let R be a noetherian valuation ring of M. Then R is an A-ring of M.*

This result may be combined with Theorem 8 to yield

THEOREM 12. *Let R be a nontrivial noetherian valuation ring of M. Then there exists a point $a \in \Omega$ such that $R = R_a = \{f \in M : f(A) \neq \infty\}$. Conversely, for each $a \in \Omega$ the ring R_a is a nontrivial noetherian valuation ring of M.*

With this result we then obtain the following theorem concerning field isomorphisms between fields of meromorphic functions.

THEOREM 13. *Let Ω_1 and Ω_2 be Riemann surfaces, where Ω_1 is noncompact. Let F_2 be a subfield of the field of functions meromorphic on Ω_2, F_2 containing the constants. Let M_1 denote the field of functions meromorphic on Ω_1, and suppose that $\theta : M_1 \to F_2$ is a field isomorphism of M_1 onto F_2. Then there exists a unique map $\phi : \Omega_2 \to \Omega_1$ such that one of the following holds:*

(1) ϕ is analytic and $\theta f = f \circ \phi$ for all $f \in M_1$,

(2) ϕ is conjugate-analytic and $\theta f = (f \circ \phi)^*$ for all $f \in M_1$.

Now if Ω is a noncompact Riemann surface, and if B is an A-ring of M, then M is the field of quotients of B. Hence Theorem 13 may be applied to ring isomorphisms between A-rings on noncompact Riemann surfaces. It results that a noncompact Riemann surface Ω is uniquely determined to within a conformal or an anti-conformal equivalence by the algebraic structure of any of the A-rings of M.

This may be considered as a generalization of the theorem of Nakai [8], who proved this result for the case where the A-ring in question is the ring A itself. It also contains the field isomorphism theorem of Iss'sa [6], the case where the A-ring involved is simply the field M.

References

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