METARECURSIVELY ENUMERABLE SETS AND ADMISSIBLE ORDINALS

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Here we describe some results which will be proved in detail in [13] and [14]. The notion of metarecursive set was introduced in [8]. Kreisel [7] reported on some of the model-theoretic deliberations which preceded the definitions of [8]. Metarecursion theory is a generalization of ordinary recursion theory from the natural numbers to the recursive ordinals. Theorems about finite sets of natural numbers are replaced by theorems about metafinite sets of recursive ordinals, some of which are infinite. Initially, metarecursive sets were defined in [8] in terms of hyperarithmetic sets, Π_1^1 sets, and notations for recursive ordinals [6], [16]; however, it later proved convenient to utilize an equation calculus devised by Kripke [9]. The purpose of Kripke's theory is to generalize recursion theory from the natural numbers to certain initial segments of the ordinals [9], [10], [11]. He calls an ordinal α admissible if the ordinals less than α have certain closure properties definable in terms of an equation calculus modeled on Kleene's. Kripke's equation calculus has numerals denoting ordinals, finitary substitution rules, and one infinitary deduction rule. If an ordinal α is admissible, then an α-recursive function f is defined by a finite system of equations: each value of f is computable using Kripke's rules, and only correct values can be so computed. It turned out that the first admissible ordinal after ω was Kleene's ω_1, the least nonrecursive ordinal, and that the metarecursive functions were the same as the ω_1-recursive functions [8], [9].

In this paper we concentrate on our first love, metarecursion theory, but we cannot resist noting, whenever appropriate, which of our results generalize to arbitrary admissible ordinals.

A set of recursive ordinals is called *regular* if its intersection with every metafinite set of recursive ordinals is metafinite. (The metafinite sets coincide with the bounded, metarecursive sets.) It was observed in [8] that there exist bounded, metarecursively enumerable sets which are not metarecursive; each such set is a constructive example of a nonregular set. It would not be unfair to say that the interesting arguments of metarecursion theory, if it is granted that

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they exist, owe their existence to the phenomenon of nonregularity. We are fond of pathology and consequently are delighted by the intricacies of Driscoll's result [2] that the metadegrees of the metarecursively enumerable sets are dense: between any two comparable ones, there is a third. On the other hand, it is always healthy to place limits on pathology. This is our purpose in the next theorem.

THEOREM 1. *Each metarecursively enumerable set has the same metadegree as some regular, metarecursively enumerable set.*

Two sets have the same metadegree [8] if each is metarecursive in the other. Theorem 1 makes it possible to lift up various results about degrees of recursively enumerable sets into metarecursion theory; it is an essential ingredient of [2]. The phenomenon of nonregularity makes it difficult, if not impossible, to lift up certain results about recursively enumerable sets. For example, we do not know if each nonmetarecursive, metarecursively enumerable set is the disjoint union of two metarecursively, enumerable sets of incomparable metadegrees. The analogous result for recursively enumerable sets is proved in [15]. The proof of Theorem 1 is inspired by Dekker's notion of deficiency set [1] and makes use of what might be called a double-deficiency set. Curiously enough, we do not know if there exists a metarecursive function f such that if ω_e is metarecursively enumerable, then $\omega_{f(e)}$ is metarecursively enumerable, regular, and of the same metadegree as ω_e.

Let α be an admissible ordinal. Following Kripke [9], we call a set α-finite if it is α-recursive and bounded by some ordinal less than α. Call a set of ordinals regular if its intersection with every α-finite set is α-finite. There exist α-recursively enumerable sets which are not regular. The proof of Theorem 1 generalizes to a proof of Theorem 1^α.

THEOREM 1^α. *Each α-recursively enumerable set has the same α-degree as some regular, α-recursively enumerable set.*

Two sets have the same α-degree if each is α-recursive in the other. The natural numbers play a crucial role in the proof of Theorem 1. The same part is played in the proof of Theorem 1^α by the ordinals less than α^*. Kripke defines α^*, the *projectum* of α, to be the least ordinal β less than or equal to α such that there exists a one-one α-recursive function whose range is bounded by β.

Let us identify the natural numbers and the finite, recursive ordinals. Then the Π_1^1 sets of natural numbers and metarecursively enummerable sets of finite, recursive ordinals coincide [8]. This coincidence strengthens the oft-mentioned analogy between recursively enumerable sets and Π_1^1 sets by making it possible to apply priority methods to Π_1^1 sets. At the moment we know of no elementary difference between recursively enumerable sets and Π_1^1 sets. For example, Friedberg's argument [**4**] can be modified to yield a maximal Π_1^1 set [**8**].

THEOREM 2. *There exist two Π_1^1 sets of natural numbers such that neither is metarecursive in the other.*

The proof of Theorem 2 of course uses the priority method, but the central trick of the argument has no counterpart that we know of in ordinary recursion theory. The Π_1^1 sets of Theorem 2, call them A and B, are defined in such a way that A and B are "incomparable" with respect to a certain subset R of the set of all metarecursive computing procedures P; this part of the proof is analogous to the usual solutions of Post's problem [**3**], [**12**]. But there is a second part of the argument which guarantees that on A and B, the procedures of R yield the same results as the procedures of P. The use of R rather than P is essential because of the requirement that all members of A and B be finite.

It is possible to regard Theorem 2 as the solution of Post's problem for Π_1^1 sets. Spector [**16**] showed that all Π_1^1 sets lie in two hyperdegrees. Theorem 2 can be extended to show that the Π_1^1 sets lie in infinitely many metadegrees.

In [**8**] it was shown that there exist two unbounded metarecursively enumerable sets whose metadegrees are incomparable. Kripke [**10**] announced that for a great variety of admissible α, Post's problem has the usual solution. We give a uniform solution of Post's problem for all admissible α.

THEOREM 3^α. *For each admissible ordinal α, there exist α-recursively enumerable sets A and B such that A is not α-recursive in B and B is not α-recursive.*

We conjecture that Theorem 2 can be generalized as follows: for each admissible ordinal α, there exist two α-recursively enumerable subsets of α^* (the projectum of α) such that neither is α-recursive in the other.

Kreisel has introduced the notion of subgeneric set of recursive ordinals. We give his precise definition in [**13**] and hope that an intuitive definition will suffice here. Let E be a finite set of equations of Kripke's equation calculus with principal function letter f and given function letter g; assume that each numeral occurring in E denotes a recursive ordinal. If we think of g as the characteristic function of some set B of recursive ordinals, then E tells us how to compute f from B. The deductions permitted by Kripke are described by ordinals. It turns out that for most B and E, the recursive ordinals do not suffice for describing all allowable computations from B using E. If B is metarecursive, they do suffice. We say B is *subgeneric* if for every E, every allowable computation from B using E can be described by a recursive ordinal. Another way of putting it is: every possible computation from B is a metafinite object. In ordinary recursion theory, every set S of natural numbers is subgeneric, since each computation from S permitted by Kripke's equation calculus is a finite object effectively given by some natural number. Kreisel asked: do there exist subgeneric sets of recursive ordinals?

THEOREM 4. *There exists a nonmetarecursive, metarecursively enumerable, subgeneric set of recursive ordinals.*

One of the reasons subgeneric sets are of interest is that vastly different notions of reducibility coincide on them; this matter is discussed by Kreisel in [7]. Theorem 4 also provides another solution to Post's problem, since it is easy to show a subgeneric set cannot be complete. Lemma 5 is important in the proof of Theorem 4. We call a set B of recursive ordinals *completely regular* if every set metarecursive in B is regular.

LEMMA 5. *A set of recursive ordinals is subgeneric if it is completely regular.*

With the help of Theorem 4 and Lemma 5, we can obtain the following strong solution of Post's problem.

THEOREM 6. *There exist two subgeneric, metarecursively enumerable sets such that neither is metarecursive in the other.*

We can generalize Theorem 4 to every admissible ordinal, but we are unable to do so with Theorem 6.

Let α be an admissible ordinal, and let R be an α-recursively enumerable set. We say R is *maximal* if the complement of R (with respect to the ordinals less than α) is not bounded by any ordinal less than α and if for each α-recursively enumerable set S, either $S-R$ or the complement of S is bounded by some ordinal less than α. In [8] it was shown that maximal, metarecursively enumerable sets exist; a slight extension of the argument used shows there are uncountably many countable admissible ordinals α such that maximal, α-recursively enumerable sets exist.

THEOREM 7. *There exists a countable admissible ordinal α with the property that no α-recursively enumerable set is maximal.*

We prove Theorem 7 by observing that no aleph-one-recursively enumerable set is maximal and then applying the Skolem-Löwenheim Theorem. We know of no uncountable admissible α such that a maximal, α-recursively enumerable set exists. There exist uncountably many α which satisfy Theorem 7, but we are unable to define any such α in a direct, elementary manner.

In [**14**] we will present axioms for recursion theory. We will see that most of the results of this paper hold for any system of ordinals and sets of ordinals satisfying the axioms. We will show that for each admissible ordinal there exists more than one recursion theory; the theory of Kripke [**9**], [**10**], [**11**] will turn out to be the minimal recursion theory for each admissible ordinal. The central feature of axiomatic recursion theory is the acceptability of recursive functions not computable in any reasonable sense. Thus Church's Thesis holds only for the "minimal" models of the axioms of recursion theory. We are presently unable to settle many questions about the classification of models of the axioms of recursion theory. With the help of Theorem 4 it is possible to give two different recursion theories on the recursive ordinals which have the same bounded, "recursive" sets, namely, the metafinite sets. It is also possible to have recursion theories containing nonconstructible sets.

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