

JORDAN ALGEBRAS AND LIE ALGEBRAS OF TYPE D_4

BY H. P. ALLEN^{1,2}

Communicated by George D. Mostow, August 16, 1965

The existence of exceptional Lie algebras of type D_4 has been known for many years. Comparatively recently, Jacobson gave a construction which produces the only known examples of exceptional D_4 's. Relatively little was known about these algebras and our main aim in this research is to study a broad class of D_4 's, called Jordan D_4 's, which includes the known exceptional examples. We approach the questions of characterizing these algebras and determining isomorphism conditions between them through the use of Galois cohomology, where the Steinberg types are taken as the standards of comparison. In what follows, unless specific mention is made, all fields are assumed to have characteristic 0 or $p \geq 5$.

Let \mathfrak{J} be an exceptional central simple Jordan algebra over a field P and let \mathfrak{k} be a cubic semisimple associative subalgebra. Then $\mathfrak{D}(\mathfrak{J}/\mathfrak{k})$, the subalgebra of the derivation algebra of \mathfrak{J} which maps \mathfrak{k} onto 0, is a Lie algebra of type D_4 . If \mathfrak{k} is a field, then $\mathfrak{D}(\mathfrak{J}/\mathfrak{k})$ is an exceptional D_4 (these are Jacobson's examples). D_4 's of this sort are called Jordan D_4 's. If $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$ is a split algebra then $\mathfrak{D}(\mathfrak{J}/\mathfrak{k})$ is a Steinberg type, and if $\mathfrak{J} = \mathfrak{h}(\mathfrak{C}_3, \gamma)$ is split while $\mathfrak{k} = \Sigma Pe_i$ is a diagonal algebra, then $\mathfrak{D}(\mathfrak{J}/\Sigma Pe_i)$ is the split D_4 over P (of classical type if $\text{char } P \neq 0$). We first determine $\text{Aut}_P \mathfrak{D}$ and $\text{Aut}_\Phi \mathfrak{D}$ where Φ is a subfield of finite codimension and Galois in P .

Let $\Gamma L_\Phi(\mathfrak{J}/\Sigma Pe_i)$ be the group of all s -semilinear transformations ($s \in G(P/\Phi)$) of \mathfrak{J} which leave ΣPe_i invariant and satisfy $N(XC) = \rho N(X)^s$, where $\rho \in P^*$ and $N(X)$ is the generic norm form on \mathfrak{J} . If $C \in \Gamma L_\Phi(\mathfrak{J}/\Sigma Pe_i)$ then " $D \rightarrow C^{-1}DC$ " $\in \text{Aut}_\Phi \mathfrak{D}$ and the correspondence $C \rightarrow "D \rightarrow C^{-1}DC"$ is an epimorphism. The kernel which we denote by K is isomorphic to $P^* \times P^* \times P^*$. If we let $\text{GL}(\mathfrak{J}/\Sigma Pe_i)$ be the subgroup of all P -linear transformations in $\Gamma L_\Phi(\mathfrak{J}/\Sigma Pe_i)$, then

$$(1) \quad I \rightarrow K \rightarrow \text{GL}(\mathfrak{J}/\Sigma Pe_i) \rightarrow \text{Aut}_P \mathfrak{D} \rightarrow I$$

is exact. (1) is not suitable for an application of Galois cohomology

¹ The research was supported by National Science Foundation grant no. NSF GP-4017.

² These results are contained in a dissertation which was submitted to Yale University in 1965 and was conducted under the supervision of Professor N. Jacobson.

since K is not in the center of $GL(\mathfrak{F}/\Sigma Pe_i)$. However we may circumvent this obstruction as follows.

Each $C \in GL_{\Phi}(\mathfrak{F}/\Sigma Pe_i)$ affects a permutation of $\{Pe_1, Pe_2, Pe_3\}$ say $P(C)$, and we let $GL(\mathfrak{F}/\{Pe_i\}_i)$ be the subgroup of $GL(\mathfrak{F}/\Sigma Pe_i)$ with $p(C) = 1$. If $Aut'_P \mathfrak{D}$ is the subgroup of $Aut_P \mathfrak{D}$ corresponding to $GL(\mathfrak{F}/\{Pe_i\}_i)$, then

$$(2) \quad I \rightarrow K \rightarrow GL(\mathfrak{F}/\{Pe_i\}_i) \rightarrow Aut'_P \mathfrak{D} \rightarrow I$$

is exact and K is in the center of $GL(\mathfrak{F}/\{Pe_i\}_i)$.

Now let \mathfrak{X} be a Φ -form of $\mathfrak{D} = \mathfrak{D}(\mathfrak{F}/\Sigma Pe_i)$, \mathfrak{F} split, and let $r \rightarrow \eta(r)$ be the corresponding pre-cocycle. Recall that $r \rightarrow \eta(r)$ is a homomorphism of $G = G(P/\Phi)$ into $Aut_{\Phi} \mathfrak{D}$ such that $\eta(r)$ is r -semilinear, and that \mathfrak{X} is the set of fixed points of \mathfrak{D} under every $\eta(r)$. If $\eta(r)$ is conjugation by $C_r \in GL_{\Phi}(\mathfrak{F}/\Sigma Pe_i)$, we let $p(r)$ be the permutation of $\{Pe_1, Pe_2, Pe_3\}$ induced by C_r . Then $r \rightarrow p(r)$ is a homomorphism of G into S_3 and the order of $p(G)$ is called the D_4 type of \mathfrak{X} and is denoted by roman numeral subscripts, e.g.,

$$D_{4I}, D_{4II}, D_{4III}, \text{ and } D_{4VI}.$$

(The exceptional algebras are those of type D_{4III} and D_{4VI} .)

It is easy to see that there are only a finite number of Steinberg D_4 's over Φ which are split by P , and more importantly, if η is any pre-cocycle of G in $Aut_{\Phi} \mathfrak{D}$, then there is a pre-cocycle μ corresponding to a Steinberg D_4 , such that $\mu^{-1}\eta$ has values in $Aut'_P \mathfrak{D}$. If we use such an μ to equip $Aut_P \mathfrak{D}$ with a G -group structure, then $Aut'_P \mathfrak{D}$ is a G -subgroup and we may define a G -action on each of the groups in (2) so that (2) becomes an exact sequence of G -groups and G -homomorphisms. The preceding remarks show that relative to this action, the 1-cocycle of G in $Aut_P \mathfrak{D}$ corresponding to \mathfrak{X} has values in $Aut'_P \mathfrak{D}$.

From (2) we obtain the exact sequence of first cohomology sets

$$(3) \quad \begin{aligned} I &= H_{\mu}^1(G, K) \rightarrow H_{\mu}^1(G, GL(\mathfrak{F}/\{Pe_i\}_i)) \rightarrow H_{\mu}^1(G, Aut'_P \mathfrak{D}) \\ &\Delta \\ &\rightarrow H_{\mu}^2(G, K) \end{aligned}$$

where the subscript μ is used to indicate the generation of the G -action and Δ is the coboundary operator. If \mathfrak{X} is an arbitrary Φ -form of \mathfrak{D} , let $[\gamma]_{\mu}$ be its corresponding 1-cocycle in the appropriate $H_{\mu}^1(G, Aut'_P \mathfrak{D})$ as above. We set $\Gamma(\mathfrak{X}) = [\gamma]_{\mu} \Delta \in H_{\mu}^2(G, K)$ and call it the 2-cohomology class of \mathfrak{X} .

The sets in (3) have simple interpretations: the elements of $H_{\mu}^1(G, GL(\mathfrak{F}/\{Pe_i\}_i))$ correspond to certain norm equivalence classes

of Φ -forms of isotopes $\mathfrak{I}^{(u)}$ of \mathfrak{I} where $u \in \Sigma Pe_i$, whereas the elements of $H_\mu^2(G, K)$ correspond to certain classes of separable associative algebras over Φ which are split by P , and play a role which generalizes the relationship between central simple associative algebras and factor sets. Finally if \mathfrak{G} is the associative subalgebra of $\text{Hom}_P(\mathfrak{I}, \mathfrak{I})$ strictly generated by \mathfrak{I} over Φ , then \mathfrak{G} is a Φ -form of $(P_\mathfrak{s} \oplus P_\mathfrak{s} \oplus P_\mathfrak{s}, J)$ where J is an involution whose restriction to each $P_\mathfrak{s}$ is the involution determined by the norm form on \mathfrak{C} . The 2-cohomology class of \mathfrak{I} , $\Gamma(\mathfrak{I})$, is then the class corresponding to \mathfrak{G} in $H_\mu^2(G, K)$.

We now have the following results.

THEOREM 1. *Let \mathfrak{I} be a Φ -form of $\mathfrak{D} = \mathfrak{D}(\mathfrak{I}/\Sigma Pe_i)$ where P/Φ is a finite dimensional Galois extension. Then \mathfrak{I} is a Jordan D_4 if and only if $\Gamma(\mathfrak{I}) = 1$.*

The condition $\Gamma(\mathfrak{I}) = 1$ is equivalent to the assertion that if \mathfrak{G} is the Φ -enveloping algebra of \mathfrak{I} , as above, then every simple component of \mathfrak{G} is a total matrix algebra over its center.

THEOREM 2. *A Lie algebra of type D_4 over a field Φ of characteristic 0 is a Jordan D_4 if and only if it is a subalgebra of an algebra of type F_4 .*

Theorem 2 follows readily from the conjugacy of subalgebras of type D_4 of a Lie algebra of type F_4 , over an algebraically closed field of characteristic 0.

THEOREM 3. *Let $\mathfrak{I} = \mathfrak{D}(\mathfrak{I}/\mathfrak{I})$ and $\mathfrak{I}^1 = \mathfrak{D}(\mathfrak{I}^1/\mathfrak{I}^1)$ be Jordan D_4 's. Then $\mathfrak{I} \cong \mathfrak{I}^1$ if and only if there is a norm equivalence between \mathfrak{I} and \mathfrak{I}^1 which carries \mathfrak{I} onto \mathfrak{I}^1 .*

Using these results we obtain

THEOREM 4. *If Φ is a local or finite field then there is a 1-1 correspondence between the set of Jordan D_4 's over Φ and the set of overfields of Φ of degree at most 3. In particular, every exceptional D_4 over Φ is a Jordan D_4 .*

Globally we can prove the following:

THEOREM 5. *Let Φ be a global field, \mathfrak{I} a Lie algebra of type D_4 over Φ . Then \mathfrak{I} is a Jordan D_4 if and only if \mathfrak{I}_{Φ_p} is a Jordan D_4 for every prime p on Φ .*

THEOREM 6. *Let Φ and \mathfrak{I} be as in the statement of Theorem 5. Then the number of primes p on Φ such that \mathfrak{I}_{Φ_p} is not a Jordan D_4 is finite; moreover none of these is real.*