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ON THE EQUATION $f^n+g^n=1$

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There is a close relationship between Fermat's last theorem and the family of solutions f and g of the functional equation

$$x^n + y^n = 1. \tag{1}$$

If, for example, S_D denotes the class of all pairs (f, g) of single valued functions f and g meromorphic in a domain D and having the additional property that, for some z_0 in D, $f(z_0)$ and $g(z_0)$ are both positive rationals, then either, for $n>2$, (1) has no solutions in S_D or Fermat's last theorem is not true.

In this note we discuss the solutions of (1) meromorphic in the complex plane. We shall call such solutions *M_c solutions*.

THEOREM 1. *For $n=2$, all M_c solutions are of the form*

$$f = \frac{2\beta(z)}{1+\beta(z)^2} \quad \text{and} \quad g(z) = \frac{1-\beta(z)^2}{1+\beta(z)^2}. \tag{2}$$

PROOF. This follows directly from a theorem on uniformization [1]. We need only note that for $n=2$, (1) is of genus zero and that the rational solution (2), with $\beta(z)=z$, maps the whole z-plane in a 1-1 manner on the Riemann surface of (1).

THEOREM 2. *For $n=3$, M_c solutions exist. One such solution is given by:*

$$\begin{aligned} f &= 4^{-1/6}(\wp')^{-1}(1 + 3^{-1/2}\cdot 4^{1/3}\wp), \\ g &= 4^{-1/6}(\wp)^{-1}(1 - 3^{-1/2}\cdot 4^{1/3}\wp), \end{aligned} \tag{3}$$

where $\wp$ is a Weierstrass $\wp$-function.

PROOF. That M_c solutions exist follows from a theorem on uniformization [1] and the fact that when $n=3$, (1) is of genus 1. The uniformization theorem assures the existence of elliptic solutions, but does not yield any simple method of constructing one. To prove that (3) is a solution we note that, with $n=3$, (1) has solutions if and only if $F^3 - G^2 = 1$ has solutions. This follows by setting $F=3^{1/2}(f-g)/(f+g)$ and $G=4^{1/3}/(f+g)$. The Weierstrass $\wp$-function satisfies a differential equation of the form

$$(y')^2 = 4y^3 - g_2 y - g_3. \tag{4}$$

When g_2 and g_3 satisfy $g_2^3 - 27g_3^2 \neq 0$, (4) is satisfied by a Weierstrass $\wp$-function whose periods depend on g_2 and g_3. Taking $g_2=0$ and $g_3=1$, we get for a particular $\wp$-function, that

$$(\wp')^2 = 4\wp^3 - 1.$$

It follows that (3) is a solution.

We know from the theory of uniformization [1] that, for $n=3$, (1) has no rational solutions. It is not known, however, what the most general solution is in this case. Thus we have

CONJECTURE 1. For $n=3$, the only M_c solutions are elliptic functions of entire functions.

M_c solutions, for $n=3$, exist only if $(1-g^3)^{1/3}$ is single valued; namely if the branch points 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$ are attained at any point either a multiple of 3 times or not at all. From the theory of Nevanlinna [2], using the notation of that theory, one sees that for any completely ramified value a, which is attained by f at any point at least n times or not at all

$$\theta(a;f) \geqq \frac{n-1}{n}\cdot$$

Since $\Sigma\theta(a;f) \leqq 2$, it follows that there exist at most $[2n/(n-1)]$ completely ramified values with the property described above. Thus conjecture 1 is included in

CONJECTURE 2. The only meromorphic functions having three completely ramified values with $n \geqq 3$ are elliptic functions of entire functions.

The above argument also gives us

THEOREM 3. *For $n>3$, M_c solutions do not exist. For $n>2$ entire solutions of* (1) *do not exist.*

The first part of the theorem also follows from Picard's uniformization theorem, once we note that, for $n>3$, the genus of 1 is greater than 1.

The second part of the theorem can also be proved by more elementary methods [**3**].

Returning to the notation at the beginning of this paper we let C be the finite complex plane and set

$$S_c^* = \{(r_1, r_2) \mid f(z) = r_1, g(z) = r_2; (f, g) \in S_c\},$$

where f and g are nonconstant and r_1 and r_2 denote positive rational numbers. We shall say further that (x, y) is a solution of (1) if x and y satisfy (1).

An immediate consequence of Theorem 3 can now be stated.

COROLLARY 1. *For $n>2$, any set of solutions $S \subset S_c^*$ is finite.*

Mordell's conjecture [**4**] that, for $n>3$ (1) has at most a finite number of possible rational solutions, is thus reduced to an interpolation problem in the theory of meromorphic functions. For it would be sufficient to show that any infinite set of solutions of (1), with $n>3$, would have to be a subset of S_c^*.

The author has also obtained some results on solutions meromorphic in a domain D. The problem of characterizing such solutions, however, is still open.

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