EXISTENCE AND UNIQUENESS OF SOLUTIONS OF THE SECOND ORDER BOUNDARY VALUE PROBLEM[1]

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The existence (and uniqueness) of a solution $y(t)$ to the two point boundary value problem

$$y''(t) + f(t, y(t), y'(t)) = 0, \tag{1}$$

$$y(a) = A, \tag{2}$$

$$y(b) = B, \tag{3}$$

is usually established by showing that the sequence of functions defined by Picard's iteration procedure converges (to $y(t)$). Picard [1], [2] showed that for the class of functions $f(t, y, y')$, which are continuous and satisfy the uniform Lipschitz condition,

$$|f(t, y, y') - f(t, x, x')| \leqq K|y - x| + L|y' - x'|, \tag{4}$$

his iteration procedure converges whenever the length of the interval $[a, b]$ is small enough. Probably the best known sufficient restriction on $b-a$ is that of Lettenmeyer [3],

$$\frac{1}{\pi^2} K(b - a)^2 + \frac{4}{\pi^2} L(b - a) < 1. \tag{5}$$

Thus if $b-a$ satisfies this inequality, then the iteration procedure converges to a function $y(t)$ which is a solution to the boundary value problem, and there is no other solution.

Although this convergence question has been investigated by a number of people over many years (see [1]–[8], for example), the maximum interval for which Picard's iteration procedure converges is still not known. And even if it were, that fact alone would not necessarily tell us anything about the maximum interval for which the boundary value problem has a unique solution, other than that it provides a lower bound. Thus Picard's method, though extremely useful for a wide class of problems, does have the one serious limitation of being applicable to only those problems for which the iteration procedure happens to converge.

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The purpose of this work is to develop an entirely different way of establishing existence and uniqueness for these boundary value problems which does not suffer from this defect. Our main results are "best possible" in the sense that if $b-a$ is greater than, or even equal to, a certain quantity which depends only upon the Lipschitz constants, then existence and uniqueness actually fail to hold in the case of at least one of the continuous functions $f(t, y, y')$ in the class under consideration.

In place of the ordinary Lipschitz condition (4), it turns out to be much more useful (and no more restrictive) to assume a set of one sided Lipschitz conditions (as was done in [9]). Namely

$$(6)\quad G_1(y-x, y'-x') \leqq f(t, y, y') - f(t, x, x') \leqq G_2(y-x, y'-x')$$

where $G_1(y, y')$ and $G_2(y, y')$ are continuous, piecewise linear functions with four coefficients K_1, K_2, L_1, L_2. (When $-L_1=L_2=L>0$ and $-K_1=K_2=K>0$ this condition (6) is equivalent to (4).)

Let $\alpha(L, K)$, $\beta(L, K)$ be the respective distances between a zero of $x(t)$ and the next and the preceding zero of $x'(t)$, $x(t)$ being any nontrivial solution of $x''+Lx'+Kx=0$. Of course $\alpha(L, K)$ and $\beta(L, K)$ can be computed explicitly.

Our main results are these:

THEOREM 1. *Let $f(t, y, y')$ be continuous and satisfy* (6). *Then the mixed boundary value problem*

$$(1)\qquad y''(t) + f(t, y(t), y'(t)) = 0,$$

$$(2)\qquad y(a) = A,$$

$$(7)\qquad y'(c) = m,$$

has exactly one solution if $0<c-a<\alpha(L_2, K_2)$.

THEOREM 2. *Let $f(t, y, y')$ be continuous and satisfy* (6). *Then the boundary value problem* (1), (2), (3) *has exactly one solution if* $0<b-a<\alpha(L_2, K_2)+\beta(L_1, K_2)$.

We call attention of the fact that in some cases either $\alpha(L_2, K_2)$ or $\beta(L_1, K_2)$ happens to be infinite, which means that the corresponding problem has a unique solution on every finite interval.

Our proofs are based upon comparisons between solutions of (1), and solutions of

$$u'' + G_i(u, u') = f(t, 0, 0), \qquad i = 1 \text{ or } 2,$$

which satisfy the same initial conditions. By permitting the K_i and L_i in (6) to be functions of t we can in many cases establish existence and uniqueness even for problems on an infinite interval, or where a singularity exists at one end of the interval. And by not requiring the Lipschitz condition to hold for *all* y, but only for y belonging to some fixed interval, we obtain an extension of the theory which includes functions $f(t, y, y')$ that are not exactly Lipschitzian.

The theorems, and certain constructions used in the proofs, also provide a basis for establishing the convergence of a number of "shooting methods" in the numerical solution of such boundary value problems. Theorem 2 generalizes the results of [10].

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