A NOTE ON LINK GROUPS

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1. **Introduction.** The purpose of this paper is to provide a sufficient geometric condition in order for a link group to be mapped homomorphically onto a free group of rank $r \leqq \mu$, where μ is the number of components of the link. Applying this together with a construction of L. Neuwirth [**3**], we obtain an extension of two results in [**3**] concerning free subgroups of link groups and solvability of link groups.

2. **Group presentations.** By a surface S of type (p, μ, r) we shall mean the disjoint union in S^3 of r tame, orientable, compact, connected 2-manifolds, each with nonempty boundary, where μ is the number of boundary components, and p is the sum of the genera of the components of S. R. H. Fox describes in [**1**] a method of obtaining for any surface of type $(p, 1, 1)$ a model S of the same embedding type which consists of a 2-cell with a number of bands attached which may be made to "lie flat" so that only one side of S is visible. This method may also be applied to a surface of type $(p, \mu, 1)$ and hence to a surface of type (p, μ, r), progressing componentwise.

If $L \subset S^3$ is a link with μ components and S is a surface of type (p, μ, r) with boundary L, then a flat model of S determines a regular projection of L from which we obtain an over presentation [**2**]

$$P_L = (x_1, \cdots, x_m : r_1, \cdots, r_m)$$

of $\pi_1(S^3 - L)$. Also, we can get a presentation

$$P_S = (z_1, \cdots, z_q : s_1, \cdots, s_v)$$

of $\pi_1(S^3 - S)$, where each z_i circles a band just once, and the relators occur where the bands cross each other. Each z_i has the form $x_{i_1} x_{i_2}^{-1}$.

3. **Main results.** Let $L \subset S^3$ be a link with μ components and genus p_0, $G = \pi_1(S^3 - L)$, F_r = free group of rank r.

THEOREM 1. *If L bounds a surface S of type (p, μ, r), then there is an epimorphism $\phi: G \rightarrow F_r$.*

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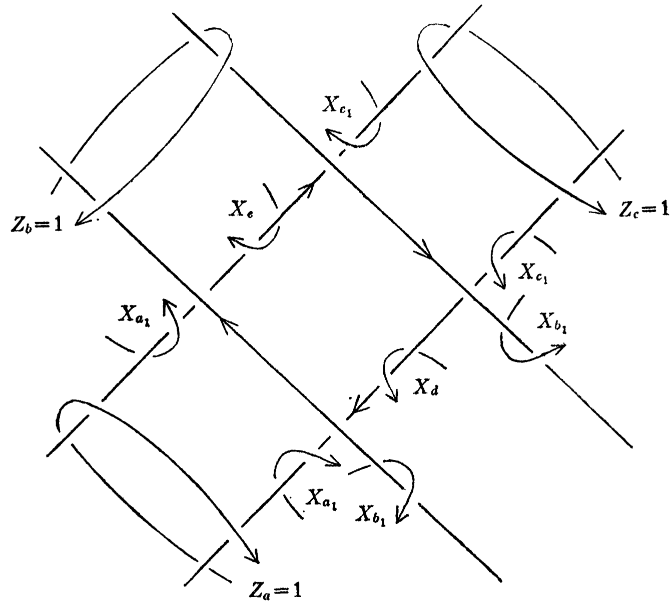

FIGURE 1.

PROOF. Without loss of generality, we may assume that no component of L is a trivial knot which is splittable from the rest of L. Let H be the image of $\pi_1(S^3-S)$ in G under the homomorphism induced by inclusion, and let $N(H)$ be the normal closure of H in G. We have the presentations P_L and P_S given in §2. A presentation P of $G/N(H)$ can be obtained by adding q relators $r_1' = z_1, \cdots, r_q' = z_q$ to P_L; thus the relations $r_j' = 1$ tell us $x_{j_1} = x_{j_2}$.

Consider first the case where $r=1$. In the regular projection of L obtained from the flat model of S, the crossing points of L occur where the bands of S cross each other, so they occur in fours (see Figure 1).

By reading relations around these crossings, we obtain $x_{a_1} = x_{c_1}$ and $x_d = x_e$. Since adjacent bands are joined along the boundary of the 2-cell (Figure 2), it follows that $x_{a_1} = x_{b_1} = x_{c_1}$, so that $x_{a_1} = x_d$. Hence $x_1 = x_2 = \cdots = x_m$ and the relators r_i become trivial since the sum of the exponents involved in each is zero. Thus P is equivalent to the presentation $(x_1, \cdots, x_m : x_1x_2^{-1}, x_2x_3^{-1}, \cdots, x_{m-1}x_m^{-1})$, so $G/N(H) \approx F_1$.

Now suppose $r>1$. If no two bands which lie on different components of S cross, then the above applies, componentwise, to give $G/N(H) \approx F_r$.

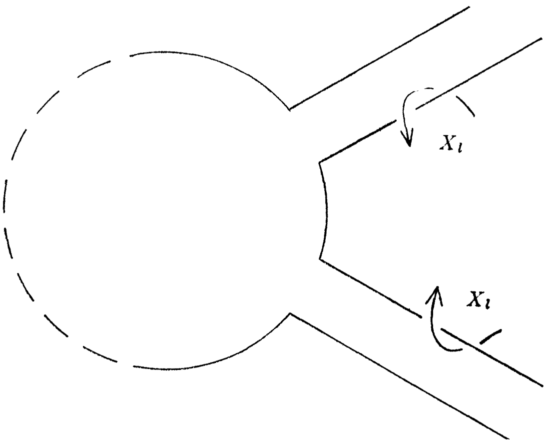

FIGURE 2.

Otherwise, consider Figure 1 where one band lies on one component of S, and the other band on another component. (At all crossings which involve only one component of S, we get all the x_i's around these four crossings equal as above, and so all the r_j's obtained at these four crossings become trivial.) We still get (as above) that $x_{a_1}=x_{c_1}$ and $x_d=x_e$. Now x_e appears and only appears in the four relators obtained at the four crossings shown; these relators are the same, viz., $x_e^{-1}x_{b_1}^{-1}x_{a_1}x_{b_1}$. Hence $x_e=x_{b_1}^{-1}x_{a_1}x_{b_1}$ in $G/N(H)$, so this relator can be eliminated together with the generator $x_e=x_d$. It follows that $G/N(H)\approx F_r$.

THEOREM 2. *If $p_0\geqq 1$ or if $p_0=0$ and $\mu>2$, then G contains a free group of rank n for any $n\leqq\infty$, and hence is not solvable.*

PROOF. It suffices to show that G contains a free group of rank $\geqq 2$. If L bounds a surface of type (p, μ, r) where $r\geqq 2$, then by Theorem 1, G can be mapped homomorphically onto F_r, and hence G contains a free group of rank $r\geqq 2$.

Now suppose L bounds no disconnected surface, and let S be a surface of type $(p_0, \mu, 1)$ with boundary L. We may now construct the covering space of S^3-L corresponding to $N(H)$ just as done in [3]. Thus $\pi_1(S)\subset G$; and

$$\operatorname{rank}\pi_1(S)=\operatorname{rank}H_1(S)=1-\chi(S)=\mu+2p_0-1\geqq 2,$$

where $\chi(S)$ is the Euler characteristic of S. Since $\pi_1(S)$ is free, the first part of the theorem follows.

Since any subgroup of a solvable group is solvable, G cannot be solvable.

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THE ENUMERATION OF LABELED TREES BY DEGREES

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1. In [1] Cayley showed that the total number of (free) trees with labeled vertices $v_1, \cdots, v_n$ is n^{n-2} by exhibiting a correspondence between them and the terms of $(v_1 + \cdots + v_n)^{n-2} v_1 v_2 \cdots v_n$. This note shows that the correspondence determines the trees of given degree specification (the degree of a point is the number of lines incident to it; the degree specification is $(k_1, k_2, \cdots)$ with k_i the number of points of degree i). More precisely, if $T(n; k_1, k_2, \cdots)$ is the number of labeled trees with degree specification $(k_1, k_2, \cdots)$ it will be shown that

$$
\begin{aligned}
T_n(x_1, x_2, \cdots) &= \sum T(n; k_1, k_2, \cdots) x_1^{k_1} x_2^{k_2} \cdots \\
&= x_1^n Y_{n-2}(f x_2 x_1^{-1}, \cdots, f x_{n-1} x_1^{-1})
\end{aligned} \qquad (1)
$$

with $f^k \equiv f_k = (n)_k = n(n-1) \cdots (n-k+1)$, and Y_n the Bell multivariable polynomial.

2. In symmetric function notation Cayley's expression is $(1)^{n-2}(1^n)$ on n variables. The multinomial theorem in symmetric function form [2, p. 43] is

$$
(1)^n = \sum \frac{n!}{1!^{k_1} \cdots n!^{k_n}} (1^{k_1} 2^{k_2} \cdots n^{k_n}), \qquad k_1 + 2k_2 + \cdots + nk_n = n.
$$

Hence, on n variables