Since any subgroup of a solvable group is solvable, G cannot be solvable.

References

1. R. H. Fox, *A quick trip through knot theory*, Topology of 3-manifolds, Prentice-Hall, Englewood Cliffs, N. J., 1962; pp. 120–167.
2. R. H. Fox and R. H. Crowell, *Introduction to knot theory*, Ginn, New York, 1963.
3. L. Neuwirth, *The algebraic determination of the genus of knots*, Amer. J. Math. 82 (1960), 791–798.

Florida State University

THE ENUMERATION OF LABELED TREES BY DEGREES

BY JOHN RIORDAN

Communicated by M. Kac, September 28, 1965

1. In [1] Cayley showed that the total number of (free) trees with labeled vertices $v_1, \cdots, v_n$ is n^{n-2} by exhibiting a correspondence between them and the terms of $(v_1+\cdots+v_n)^{n-2}v_1v_2\cdots v_n$. This note shows that the correspondence determines the trees of given degree specification (the degree of a point is the number of lines incident to it; the degree specification is $(k_1, k_2, \cdots)$ with k_i the number of points of degree i). More precisely, if $T(n; k_1, k_2, \cdots)$ is the number of labeled trees with degree specification $(k_1, k_2, \cdots)$ it will be shown that

$$\begin{aligned} T_n(x_1, x_2, \cdots) &= \sum T(n; k_1, k_2, \cdots) x_1^{k_1} x_2^{k_2} \cdots \\ &= x_1^n Y_{n-2}(f x_2 x_1^{-1}, \cdots, f x_{n-1} x_1^{-1}) \end{aligned} \tag{1}$$

with $f^k \equiv f_k = (n)_k = n(n-1)\cdots(n-k+1)$, and Y_n the Bell multivariable polynomial.

2. In symmetric function notation Cayley's expression is $(1)^{n-2}(1^n)$ on n variables. The multinomial theorem in symmetric function form [2, p. 43] is

$$(1)^n = \sum \frac{n!}{1!^{k_1}\cdots n!^{k_n}} (1^{k_1}2^{k_2}\cdots n^{k_n}), \qquad k_1 + 2k_2 + \cdots + nk_n = n.$$

Hence, on n variables

$$(2)\quad (1)^{n-2}(1^n) = \sum \frac{(n-2)!}{1!^{k_1}\cdots(n-2)!^{k_{n-2}}}(1^{n-k}2^{k_1}3^{k_2}\cdots(n-1)^{k_{n-2}})$$

with $k=k_1+k_2+\cdots+k_{n-2}$. In the symmetric function $(1^{n-k}2^{k_1}\cdots(n-1)^{k_{n-2}})$ there are

$$\frac{n!}{(n-k)!k_1!\cdots k_{n-2}!}$$

terms in which $n-k$ variables are of degree 1, and k_i variables are of degree $i+1$. Hence

$$\begin{aligned}(1)\,T_n(x_1,x_2,\cdots) &= \sum \frac{(n-2)!(n)_k x_1^n}{k_1!\cdots k_{n-2}!}\left(\frac{x_2}{1!x_1}\right)^{k_1}\cdots\left(\frac{x_{n-1}}{(n-2)!x_1}\right)^{k_{n-2}}\\ &= x_1^n Y_{n-2}(fx_2x_1^{-1},\cdots,fx_{n-1}x_1^{-1}),\qquad f^k\equiv f_k=(n)_k\end{aligned}$$

as stated above; the notation follows [4].

3. Some special cases of (1) are worth noting. First the enumerator by number of points of degree 1 (end points) is $T_n(x,1,1,\cdots)$. But

$$Y_n(x,x,\cdots) = a_n(x) = \sum_{k=0}^{n} S(n,k)x^k$$

with $S(n,k)$ the Stirling number of the second kind. Hence

$$\begin{aligned}T_n(x,1,\cdots) &= x^nY_{n-2}(fx^{-1},fx^{-1},\cdots,fx^{-1})\\ &= \sum_{k=0}^{n-2} S(n-2,k)f_kx^{n-k} \qquad (3)\\ &= \sum_{k=2}^{n} S(n-2,n-k)(n)_{n-k}x^k\end{aligned}$$

which is given by A. Rényi in [3].

Next the enumerator by number of points of degree 2 is

$$T_n(1,x,1,\cdots) = Y_{n-2}(fx,f,\cdots,f).$$

Since

$$\exp uY(fg_1,fg_2,\cdots) = \exp f(ug_1+u^2g_2/2!+\cdots),\qquad Y^k\equiv Y_k,\ f^k\equiv f_k,$$

it follows that

$$\begin{aligned}\exp uY(fx,f,\cdots) &= \exp f(e^u-1-u+ux)\\ &= \exp u(b(f)+fx),\qquad b^k(f)\equiv b_k(f),\end{aligned}$$

where $b_n(s)$ is the associated Stirling number polynomial (enumerating permutations by number of nonunitary ordered cycles) given in [4, p. 77]. Hence

$$T_n(1, x, 1, \cdots) = (b(f) + fx)^{n-2}, \qquad f^k \equiv f_k = (n)_k$$

$$= \sum_{k=0}^{n-2} \binom{n-2}{k} x^k \sum_{j=0} b_{n-2-k,j}(n)_{k+j} \tag{4}$$

with numbers b_{nj} defined by $b_n(s) = \sum b_{nj}s^j$ (a short table appears in [4]).

References

1. A. Cayley, *A theorem on trees*, Quart, J. Math. **23** (1889), 376–378 = Collected Papers, Cambridge, 1897, Vol. 13, pp. 26–28.

2. P. A. MacMahon, *Combinatory analysis*, Vol. I, Cambridge Univ. Press, London, 1915.

3. A. Rényi, *Some remarks on the theory of trees*, Magyar Tud. Akad. Mat. Kutató Int. Közl. **4** (1959), 73–85.

4. John Riordan, *An introduction to combinatorial analysis*, Wiley, New York, 1958.

Bell Telephone Laboratories, Inc., Murray Hill, New Jersey