A CHARACTERIZATION OF THE EUCLIDEAN SPHERE

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1. Introduction. Let $M$ be a connected Riemannian manifold of dimension $n$, $C_0(M)$ its largest connected group of conformal transformations and $I_0(M)$ its largest connected group of isometries. In an earlier paper [2], one of the authors and S. Kobayashi established the following result:

**Theorem 1.** A compact homogeneous Riemannian manifold for which $C_0(M) \neq I_0(M)$ and $n > 3$ is globally isometric with a sphere.\(^2\)

In the final step of the proof of this theorem the following statement, which is by no means easy to establish, was utilized:

**Proposition 1 (YANO-NAGANO [6]).** A complete Einstein space for which $C_0(M) \neq I_0(M)$ and $n > 2$ is globally isometric with a sphere.

Without this fact it was shown that the simply connected Riemannian covering of $M$ is globally isometric with a sphere. Using this statement, an elementary proof of Theorem 1, i.e. a proof which does not use Proposition 1, is given (see Proposition 4).

All other results in this direction employ Proposition 1 in the final analysis. We list several of these:

**Proposition 2 (NAGANO [4]).** A complete Riemannian manifold with parallel Ricci tensor for which $C_0(M) \neq I_0(M)$ and $n > 2$ is globally isometric with a sphere.

This generalizes Proposition 1.

**Proposition 3 (LICHNEROWICZ [3]).** Let $M$ be a compact Riemannian manifold of dimension $n > 2$ whose scalar curvature $R$ is a positive constant and for which $\text{trace } Q^2 = \text{const.}$ where $Q$ is the Ricci operator (see [1, p. 87]). Then, if $C_0(M) \neq I_0(M)$, $M$ is globally isometric with a sphere.

This generalizes Theorem 1 and Proposition 2.

In §4, Proposition 1 will be generalized. Denote the Lie algebra of

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\(^2\) The first part of the proof of Theorem 1 appears in a previous paper published in the Amer. J. Math 84 (1962), 170–174 by S. I. Goldberg and S. Kobayashi entitled The conformal transformation group of a compact Riemannian manifold.
Let $X \in C_0(M)$ and $\xi$ be the covariant form of $X$ defined by duality by the Riemannian metric $\langle \cdot, \cdot \rangle$ of $M : \xi = \langle X, \cdot \rangle$. Let $C^*_\omega(M) = \{ \xi | \xi = \langle X, \cdot \rangle, X \in C_0(M) \}$ and denote by $d$ and $\delta$ the differential and codifferential operators of de Rham and Hodge. Then [cf. M. Obata and K. Yano, C. R. Acad. Sci. Paris 260 (1965), 2698–2700].

**Theorem 2.** Let $M$ be a compact Riemannian manifold of dimension $n > 3$ for which $R = \text{const.}$ and $C_0(M) \neq I_0(M)$. If $d \delta C^*_\omega(M)$ is an invariant subspace of $Q$, then $M$ is globally isometric with a sphere.

This theorem most completely answers the question raised in [2], namely,

Is a compact manifold of dimension $n > 2$ with constant (positive) scalar curvature for which $C_0(M) \neq I_0(M)$ isometric with a sphere?

Observe that Proposition 1 is an easy consequence of Theorem 2.

2. Isometries and conformal fields. If $T$ is an isometry of the unit sphere $S^n$ in $E^{n+1}$, then $T$ may be viewed as an orthogonal linear transformation of $E^{n+1}$ restricted to $S^n$. It is clear that any such isometry will map Killing fields into Killing fields and constant conformal fields ($d\phi = \langle X, \cdot \rangle$) into constant conformal fields. Thus if a conformal field is invariant under $T$ so are its constant and Killing parts. It follows that if $T$ leaves a non-Killing conformal field invariant then it has a fixed point, namely $N||N|| \in S^n$, where $N$ is a constant field in $E^{n+1}$ and $N - \langle N, P \rangle P$ ($P \in S^n$) is the constant part of $V$.

3. Conformal fields on a manifold of positive constant curvature. If $M$ is a compact Riemannian manifold with constant positive curvature then the nature of the conformal group of $M$ does not change if we normalized the curvature so that it is 1. Thus, $S^n$ is the simply connected covering Riemannian manifold of $M$. If $M$ has a non-Killing conformal vector field $V$ then this vector field may be lifted to a non-Killing conformal vector field $\overline{V}$ on $S^n$. Moreover, $\overline{V}$ is invariant by the deck transformations of the covering space $S^n \rightarrow M$. But only the identity deck transformation can have a fixed point, and since a deck transformation is an isometry we have from §2 that there are no deck transformations except the identity. This proves the following special case of Proposition 1:

**Proposition 4.** If a compact Riemannian manifold of positive constant curvature admits a non-Killing conformal vector field then it is globally isometric with a sphere.

Since the above argument clearly works for $n = 2$, we have
Corollary. The real projective plane does not admit a non-Killing conformal vector field.

4. Conformal fields on manifolds of constant scalar curvature. We sketch the proof of Theorem 2. Let $\xi = d\phi$ be an element of $C^*_0(M)$. Then, $Qd\delta\xi = d\delta Q\xi$. Conversely, suppose $d\delta C^*_0(M)$ is an invariant subspace of $Q$. Then, there exists a $\xi \in C^*_0(M)$ such that $d\delta\xi$ is an eigenvector of $Q$, that is $Qd\delta\xi = (R/n)d\delta\xi \in C^*_0(M)$. Moreover, since $\Delta d\xi = (R/(n-1))d\xi$ (see [1, p. 264]),

$$d\delta\xi = \frac{R}{n-1}\xi + \langle Y, \cdot \rangle$$

where $Y$ is a Killing field. That this can only hold if $M$ has constant curvature is a consequence of the following:

Lemma. Let $M$ be a compact Riemannian manifold on which there is a nonconstant function $\phi: M \to \mathbb{R}$ whose gradient $\xi = d\phi \in C^*_0(M)$. Then, there are no nonzero tensors of the type $(r, s)$, $0 < 2(s-r) < n$ invariant under $X$ where $\xi = \langle X, \cdot \rangle$.

The proof of this lemma is intended for a subsequent paper.

Setting

$$T(A, B) = \mathcal{R}(A, B) - \frac{R}{n} \langle A, B \rangle,$$

where $\mathcal{R}$ is the Ricci tensor, it can be shown that $\theta(\xi)T = 0$. Since the Weyl conformal curvature tensor is invariant under $X$, we see by the lemma that $M$ is conformally flat. However, since $\theta(\xi)T$ vanishes, a further application of the lemma gives $T = 0$, that is $M$ is an Einstein space. But a conformally flat Einstein space has constant curvature, and so by Proposition 4, $M$ is globally isometric with a sphere. This proves Theorem 2 and generalizes Proposition 1.

References