HIGHER PRODUCTS

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W. S. Massey has defined a class of higher order cohomology operations of several variables, the higher products [2]. In this paper, we shall present a relativized definition of the higher products. We shall go on to list some of the algebraic and functorial properties of these operations. Finally, we shall describe a related cohomology operation of one variable. In certain cases, the latter operation can be computed in terms of primary Steenrod operations.

1. Notation and definitions. Throughout this paper, let $X$ be a topological space and let $(X_i, A_i)$ be pairs of subspaces of $X$, for $i = 1, \ldots, k$, such that $\bigcup_{i=1}^{r} A_i \subseteq \bigcap_{i=1}^{r} X_i$. Furthermore, for $1 \leq i, j \leq k$, assume that the triads $(X_i, A_i, A_j)$ are excisive in the singular cohomology theory. This condition is satisfied if each $X_i$ and $A_i$ are open in $X$ or if $X$ is a CW complex and each $X_i$ and $A_i$ are subcomplexes. Let $u_1, \ldots, u_k$ be cohomology classes in the singular cohomology groups $H^p(X, A_i)$, respectively, where the coefficients are in a fixed commutative ring $R$ with identity. Finally, let $p(i, j) = \sum_{r=1}^{j} p_r - 1$ and $(X, A) = (\bigcap_{i=1}^{r} X_i, \bigcup_{i=1}^{r} A_i)$.

Under certain conditions, we may define the $k$-fold product $\langle u_1, \ldots, u_k \rangle$. Our definition shall be similar to the provisional definition of Massey [2].

DEFINITION 1. A defining system for $\langle u_1, \ldots, u_k \rangle, A$, is a set of singular cochains $(\alpha_{i,j})$, for $1 \leq i \leq j \leq k$ and $(i, j) \neq (1, k)$, satisfying the conditions:

1. $\alpha_{i,j} \in C^{p(i,j)+1}(\bigcap_{r=1}^{j} X_r, \bigcup_{r=1}^{j} A_r)$,
2. $\alpha_{i,j}$ is a cocycle representative of $u_i$, $i = 1, \ldots, k$ and $(i, j) \neq (1, k)$, and
3. $\delta\alpha_{i,j} = \sum_{r=1}^{j-1} (-1)^{(j+1-r)p(i,r)} \alpha_{i,r} \alpha_{r+1,j}$.

The related cocycle of $A$ is the singular cocycle of $C^*(X, A)$

\[ \sum_{r=1}^{k-1} (-1)^{(k+1-r)p(1,r)} a_{1,r} a_{r+1,k} \]

DEFINITION 2. The $k$-fold product $\langle u_1, \ldots, u_k \rangle$ is said to be defined if there is a defining system for it. If it is defined, then $\langle u_1, \ldots, u_k \rangle$ consists of all classes $w \in H^p(X, A)$ for which there exists a defining system $A$ whose related cocycle represents $w$.

If $k = 2$, then the higher product $\langle u_1, u_2 \rangle$ is the ordinary cup product.

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u_1 u_2. If \( k = 3 \), then \( \langle u_1, u_2, u_3 \rangle \) is defined if and only if the cup products \( u_1 u_2 = 0 \) and \( u_2 u_3 = 0 \). In this case the related cocycles are of the form 
\[
a_{12} a_{23} - (-1)^m a_{13} a_{23}.
\]
This is a secondary operation, the Massey triple product as defined in [4].

The \( k \)-fold product is a \( \langle k - 1 \rangle \)-order cohomology operation of \( k \) variables. In order for \( \langle u_1, \ldots, u_k \rangle \) to be defined, it is necessary that the \( (k - 2) \)-order operations \( \langle u_1, \ldots, u_{k-1} \rangle \) and \( \langle u_2, \ldots, u_k \rangle \) be defined and contain the zero element. In general this condition is not sufficient. There must exist defining systems \( A' \) and \( A'' \) for \( \langle u_1, \ldots, u_{k-1} \rangle \) and \( \langle u_2, \ldots, u_k \rangle \) respectively, for which not only do the related cocycles of each cobound but also \( a'_{i,j} - a''_{i,j} \) for \( 1 < i \leq j < k \). In this case, we say that \( \langle u_1, \ldots, u_{k-1} \rangle \) and \( \langle u_2, \ldots, u_k \rangle \) vanish simultaneously.

2. Properties. We take the position that the higher products are analogous to the cohomology cup product. The properties listed below are generalizations of well-known relations satisfied by the cup product.

2.1. Naturality. For \( i = 1, \ldots, k \), let \( \langle Y_i, B_i \rangle \) be pairs of subspaces of the topological space \( Y \) satisfying the conditions of §1. Let \( g: Y \to X \) be a continuous map such that the image of \( \langle Y_i, B_i \rangle \) under \( g \) is contained in \( \langle X_i, A_i \rangle \) and denote by \( g_*: \langle Y_i, B_i \rangle \to \langle X_i, A_i \rangle \) the induced map. Also, with \( \langle Y, B \rangle = (\cap_{i=1}^k Y_i, \cup_{i=1}^k B_i) \), let \( \tilde{g}: \langle Y, B \rangle \to \langle X, A \rangle \) be the induced map. If \( \langle u_1, \ldots, u_k \rangle \) is defined, then so is 
\[
\langle g^* u_1, \ldots, g^* u_k \rangle \subset \langle g^* u_1, \ldots, g^* u_k \rangle.
\]

2.2. Scalar multiplication. Assume that the product \( \langle u_1, \ldots, u_k \rangle \) is defined. Then \( \langle u_1, \ldots, x u_t, \ldots, u_k \rangle \) is defined for any \( x \in \mathbb{R}, t = 1, \ldots, k \) and \( x(u_1, \ldots, u_k) \subset \langle u_1, \ldots, x u_t, \ldots, u_k \rangle \).

2.3. Coboundary formula. For some \( t = 1, \ldots, k \), assume that \( \langle X_t, A_t \rangle = (B, C) \) and \( \langle X_i, A_i \rangle = (Y, C) \) for \( i \neq t \), where \( (Y, B, C) \) is a triple of topological spaces. If \( \langle u_1, \ldots, u_t, \ldots, u_k \rangle \) is defined as a subset of \( H^{p(t,k)+t}(B, C) \), then \( \langle u_1, \ldots, \delta u_t, \ldots, u_k \rangle \) is defined as a subset of \( H^{p(t,k)+t}(Y, B) \) and
\[
\delta \langle u_1, \ldots, u_t, \ldots, u_k \rangle \subset (-1)^m \langle u_1, \ldots, \delta u_t, \ldots, u_k \rangle
\]
with \( m = \sum_{t=1}^k p_t + k \).

2.4. Loop suspension. Let \( \pi: PX \to X \) be the path loop fibration over \( X \). Then \( E_\Delta = \pi^{-1}(A) \) is the space of paths in \( X \) starting from the base point and ending in \( A \). The relative loop suspension homomorphism \( \sigma: H^n(X, A) \to H^{n-1}(E_\Delta) \) is defined as the composite map
\[
H^n(X, A) \xrightarrow{\pi^*} H^n(PX, E_\Delta) \xrightarrow{\delta} H^{n-1}(E_\Delta).
\]
Assume that \( \langle u_1, \cdots, u_k \rangle \) is defined as a subset of \( H^p(1,k)+2(X, A) \). Then \( \sigma(u_1, \cdots, u_k) \) is the subset of \( H^p(1,k)+1(E_A) \) consisting solely of the zero element.

2.5. Associativity. Let \( \langle u_1, \cdots, u_k \rangle \) be defined as a subset of \( H^p(1,k)+2(X, A) \) and let \( v \in H^q(X', A') \), where \( (X', A') \) is also a pair of subspaces of \( X \). Then the \( k \)-fold product \( \langle u_1, \cdots, u_k \rangle \) is defined for each \( t = 1, \cdots, k \) as a subset of \( H^p(1,k)+q+2(X \cap X', A \cup A') \) and satisfies the relations

\[
\langle u_1, \cdots, u_k \rangle \subset \langle u_1, \cdots, u_k, v \rangle,
\]

\[
\sigma(\langle u_1, \cdots, u_k \rangle) \subset (-1)^{k^2} \langle vu_1, \cdots, u_k \rangle
\]

and

\[
\langle u_1, \cdots, u_t, v, u_{t+1}, \cdots, u_k \rangle \cap \langle u_1, \cdots, u_t, vu_{t+1}, \cdots, u_k \rangle \neq \emptyset.
\]

These relations may be interpreted as equalities modulo the sum of the indeterminacies.

2.6. Symmetry. Assume that the higher product \( \langle u_1, \cdots, u_k \rangle \) is defined. Then the symmetric product \( \langle u_k, \cdots, u_1 \rangle \) is also defined and \( \langle u_1, \cdots, u_k \rangle = (-1)^n \langle u_k, \cdots, u_1 \rangle \) with \( n = \sum_{1 \leq r < s \leq k} p_r p_s + (k-1)(k-2)/2 \).

2.7. Permutability. Assume that all the \( k \)-fold products \( \langle u_1, \cdots, u_k \rangle \) are defined simultaneously as subsets of \( H^p(1,k)+2(X, A) \). Then there are classes \( w_t \in \langle u_1, \cdots, u_{t-1} \rangle \), for \( t = 1, \cdots, k \), such that \( \sum_{t=1}^k (-1)^{t(k+1)+\pi(t)} w_t = 0 \), where \( \pi(1) = 0 \) and \( \pi(t) = (p_1 + \cdots + p_{t-1})(p_{t+1} + \cdots + p_k) \) for \( t > 1 \).

The proofs of these formulas and relations are computational in nature. For the proof of 2.5, we use the \( u_1 \)-product of Steenrod [3] and a formula of G. Hirsch [1]. The formulas 2.6 and 2.7 require the use of a set of “commuting” chain homotopies which we may construct by means of the acyclic model theorem.

3. The operation \( \langle u \rangle^k \). If we assume that \( u_1 = u_2 = \cdots = u_k = u \in H^m(X, A) \), then we can define a related higher order cohomology operation \( \langle u \rangle^k \) with less indeterminacy.

DEFINITION 1'. A defining system for \( \langle u \rangle^k \), \( A^* \), is a set of singular cochains \( (a_n) \), for \( n = 1, \cdots, k-1 \), satisfying the conditions:

\[
\begin{aligned}
(3.1) & \quad a_n \in C^n(m-1)+1(X, A), \\
(3.2) & \quad a_1 \text{ is a cocycle representative of } u, \text{ and} \\
(3.3) & \quad \delta a_n = \sum_{r=1}^{n-1} (-1)^{rn}(m-1) a_r a_{n-r}.
\end{aligned}
\]

The related cocycle of \( A^* \) is the singular cocycle of \( C^*(X, A) \)

\[
\begin{aligned}
(3.4) & \quad \sum_{r=1}^{k-1} (-1)^{rk}(m-1) a_r a_{k-r}.
\end{aligned}
\]
DEFINITION 2'. The operation \( (u)^k \) is said to be defined if there is a defining system for it. If it is defined, then \( (u)^k \) consists of all classes \( w \in H^k(X; A) \) for which there exists a defining system \( A^* \) whose related cocycle represents \( w \).

If \( (u)^k \) is defined, then so is the \( k \)-fold product \( (u, \ldots, u) \) and \( (u)^k \subset (u, \ldots, u) \). Also \( (u)^k \) is defined if and only if \( (u)^k \) is defined and contains the zero class.

Let \( p \) be an odd prime and let \( \beta \) be the Bockstein operator associated with the exact sequence of coefficient groups \( 0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to \mathbb{Z}_p \to 0 \). Furthermore, let \( P^m \) be the Steenrod \( p \)th power operation,

\[
P^m: H^*(X; \mathbb{Z}_p) \to H^{*+2m(p-1)}(X; \mathbb{Z}_p).
\]

**Theorem A.** If \( u \in H^{2m+1}(X; \mathbb{Z}_p) \), then \( (u)^p \) is defined as a single class in \( H^{2mp+2}(X; \mathbb{Z}_p) \) and \( (u)^p = -\beta P^m u \).

If \( u \) is a one-dimensional class \( (\text{mod } p) \) for any prime \( p \), then we may completely characterize the operation \( (u)^k \) by the following theorem.

**Theorem B.** Let \( i \in H^1(\mathbb{Z}_p; \mathbb{Z}_p) \) be the mod \( p \) reduction of the fundamental class \( i_n \) of \( H^1(\mathbb{Z}_p; \mathbb{Z}_p) \). Then \( (i)^p \) is defined as the single class \(-\beta_n i_n \in H^2(\mathbb{Z}_p; \mathbb{Z}_p)\), where \( \beta_n \) is the Bockstein coboundary operator associated with the exact sequence of coefficient groups

\[
0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to \mathbb{Z}_p \to 0.
\]

**Bibliography**


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