EXAMPLES IN HELSON SETS

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A compact subset P of a locally compact abelian group G is said to be a *Kronecker set* in G [1, p. 97] if every continuous unimodular function on P is uniformly approximable on P by continuous characters of G. P is a *Helson set* [1, pp. 114–115] if for some $\epsilon > 0$ and each $\mu \in M(P)$:

$$(H, \epsilon) \quad \epsilon \|\mu\| \leqq \sup_{\gamma \in \Gamma} \left| \int_G \gamma(x) d\mu(x) \right|, \qquad \|\mu\| = |\mu|(P),$$

Γ being the dual of G.

If P is a Kronecker set in G, P satisfies $(H, 1)$ by [1, Lemma 5.5.1]. It was asked in [1] whether $(H, 1)$ implies that P is a Kronecker set. Wik [2] constructed a class of counter-examples in the real line; in this note a different type of construction is announced.

Let X be a compact Hausdorff space and U the (abstract) group of continuous unimodular functions on X, Γ a subgroup of U which separates the points of X. Then X is embedded as a topological subspace of $\hat{\Gamma}$ and is a Kronecker set in $\hat{\Gamma}$ if and only if Γ is uniformly dense in U. We give below two examples in which Γ is a proper closed subgroup of U but for which $(H, 1)$ holds for measures in X.

(a) X is the 1-torus and Γ the group of functions with winding number, or degree, zero. In this case the Kronecker condition holds on the complement of any arc, so $(H, 1)$ holds.

(b) X is the unit interval $[0, 1]$ and Γ is the set of all functions e^{if}, f real and $\int_0^1 f\,dx = 0$. In this case $U = \Gamma \cdot \mathbf{C}$, $\mathbf{C}$ being the subgroup of constant functions.

In (a) and (b) the groups Γ have the form $\exp^{iH}$, where H is an additive subgroup of the real continuous functions on X. In each case H contains a dense subgroup H_1 algebraically isomorphic to $Z \oplus Z \oplus Z \oplus \cdots$; the exponential mapping is an isomorphism onto Γ. In (a) H_1 is the subgroup of trigonometric polynomials with coefficients in $Z + \sqrt{2}Z$; in (b) one uses the same coefficients with the generators $\{x^n - 1/(n+1) : n \geqq 1\}$. Using the smaller subgroups of U determined by these subspaces we can embed $X \to T^\omega$ and have the same phenomenon in regard to measures in X. In view of Theorem

5.2.9 of [1] this is probably the simplest group in which the torus and line can be Helson sets. Of course in neither case does the embedding have any connection at all with the usual group operations in X.

REFERENCES

1. W. Rudin, *Fourier analysis on groups*, Interscience, New York, 1962.

2. I. Wik, *Some examples of sets with linear independence*, Ark. Mat. **5** (1964), 207–214.

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