FIRST ORDER PROPERTIES OF PAIRS OF CARDINALS

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We consider models of a countable first order logic L with an identity symbol and predicate symbols $U, P_0, P_1, \cdots, U$ being unary. A model $\mathfrak{A} = \langle A, U_{\mathfrak{A}}, P_{0\mathfrak{A}}, \cdots \rangle$ for L is said to be a *two-cardinal model* if A is infinite and the power of $U_{\mathfrak{A}}$ is less than the power of A. By a *set of axioms for two-cardinal models* we mean a set Σ of sentences of L such that $\mathfrak{A}$ is a model of Σ if and only if there exists a two-cardinal model which is elementarily equivalent to $\mathfrak{A}$. Using results of Fuhrken [1], Vaught [4] proved the following theorem.

THEOREM (VAUGHT). *There is a set of axioms for two-cardinal models. If the language L is recursive, then there is a recursive set of axioms for two-cardinal models.*

We say that L is recursive if the number of argument places of the symbol P_n is a recursive function of n. Vaught's proof depends on the fact that if Σ^* is a recursive set of sentences in an extension L^* of the language L, then there is a recursive set Σ of sentences of L such that Σ and Σ^* have exactly the same consequences in L. In principle his proof can be used to construct a particular set of axioms for two-cardinal models, but the set seems to be so complicated that in practice one cannot easily tell whether or not a given sentence belongs to it. Vaught has proposed the problem of finding a simple set of axioms for two-cardinal models. The author heard about Vaught's problem through Dana Scott.

In this note we shall give a particular set of axioms for two-cardinal models which is simple enough to be written down as a fairly short axiom scheme. Our theorem was stated without proof in [2]. Let the individual variables of L be v_i, x_i, y_i, z_i, where $i = 0, 1, 2, \cdots$.

THEOREM 1. *A set of axioms for two-cardinal models is given by the set Γ of all sentences of the form*

$$\exists v_0 \forall x_0 \exists y_0 z_0 \cdots \forall x_n \exists y_n z_n$$

$$(*) \qquad \left[\bigwedge_{i=0}^{n} v_0 \neq y_i \; \& \bigwedge_{i,j=0}^{n} (U(x_j) \;\&\; x_i = z_j \rightarrow y_i = x_j) \right.$$
$$\left. \& \bigwedge_{j=0}^{m} (\phi_j(x_0, \cdots, x_n) \rightarrow \phi_j(y_0, \cdots, y_n)) \right].$$

There is one instance of the scheme (*) for each n and each finite sequence of formulas $\phi_0, \cdots, \phi_m$ of L with the free variables $x_0, \cdots, x_n$.

It is obvious that the set Γ of sentences is recursive provided that the language L is recursive. To prove Theorem 1, we shall use a lemma of Vaught, which is proved in Morley and Vaught [3, p. 55]. We use the standard notations $\mathfrak{A}\cong\mathfrak{B}$, $\mathfrak{A}\equiv\mathfrak{B}$, $\mathfrak{A}\prec\mathfrak{B}$, to mean that $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$, $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$, and $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$ (see, for example, [3]).

LEMMA (VAUGHT). *For each model $\mathfrak{A}$ for L, the following two conditions are equivalent:*

(i) *There is a two-cardinal model $\mathfrak{B}$ such that $\mathfrak{B}\equiv\mathfrak{A}$.*

(ii) *There exist countable models $\mathfrak{B}$, $\mathfrak{C}$, such that $\mathfrak{B}\equiv\mathfrak{A}$, $\mathfrak{C}\prec\mathfrak{B}$, $\mathfrak{C}\neq\mathfrak{B}$, $\mathfrak{C}\cong\mathfrak{B}$, and $U_{\mathfrak{C}}=U_{\mathfrak{B}}$.*

We now prove Theorem 1. First the easy direction. We let $\mathfrak{A}$ be elementarily equivalent to a two-cardinal model and prove that $\mathfrak{A}$ is a model of Γ. Consider a sentence ψ of the form (*) in Γ. Let $\mathfrak{B}$, $\mathfrak{C}$ be as in part (ii) of the lemma and let f be an isomorphism from $\mathfrak{B}$ to $\mathfrak{C}$. For all ϕ_i and all $b_0, \cdots, b_n\in B$, the following are equivalent:

$$b_0, \cdots, b_n \text{ satisfies } \phi_i \text{ in } \mathfrak{B};$$

$$fb_0, \cdots, fb_n \text{ satisfies } \phi_i \text{ in } \mathfrak{C};$$

$$fb_0, \cdots, fb_n \text{ satisfies } \phi_i \text{ in } \mathfrak{B}.$$

We shall use the fact that the first line above implies the third line. To show that ψ holds in $\mathfrak{B}$, we find an element v_0 in B and functions $y_i(x_0, \cdots, x_i)$, $z_i(x_0, \cdots, x_i)$, $i=0, \cdots, n$ on B such that the inner part of ψ holds in $\mathfrak{B}$ for all $x_0, \cdots, x_n$ in B. Take for v_0 any element of $B-C$. Let $y_i(x_0, \cdots, x_i)=f(x_i)$. If $U(x_i)$, let $z_i(x_0, \cdots, x_i)=f^{-1}(x_i)$, and otherwise choose z_i arbitrarily. These choices of v_0, y_i, z_i show that ψ holds in $\mathfrak{B}$ and thus in $\mathfrak{A}$. Therefore $\mathfrak{A}$ is a model of ψ.

We now prove the converse. Assume $\mathfrak{A}$ is a model of Γ. We extend the language L to a language L^* by adding a new individual constant c and function symbols F_n, G_n with $n+1$ argument places, $n=0, 1, 2, \cdots$. Let Γ^* be the set of all the sentences below:

(1) $\forall x_0 \cdots x_n,\ c\neq F_n(x_0, \cdots, x_n)$.

(2) $\forall x_0 \cdots x_n,\ (U(x_j)\ \&\ x_i=G_j(x_0, \cdots, x_j)\rightarrow F_i(x_0, \cdots, x_i)=x_j)$.

(3) $\forall x_0 \cdots x_n,\ [\phi(x_0, \cdots, x_n)\rightarrow\phi(F_0(x_0), \cdots, F_n(x_0, \cdots, x_n))]$.

The scheme (1) contains one sentence for each n, (2) contains a sentence for each n and each $i, j\leqq n$, while (3) contains one sentence for each n and each formula $\phi(x_0, \cdots, x_n)$ of the original language L. Since $\mathfrak{A}$ is a model of Γ, it follows that for each finite subset $\Gamma_0^*\subset\Gamma^*$

the model $\mathfrak{A}$ can be expanded to a model $(\mathfrak{A}, c, F_0, \cdots, G_0, \cdots)$ of Γ_0^*. Let Δ be the set of all sentences of L which hold in $\mathfrak{A}$. Then the set of sentences $\Delta \cup \Gamma^*$ is finitely satisfiable. By the compactness and Löwenheim-Skolem theorems, $\Delta \cup \Gamma^*$ has a countable model $(\mathfrak{B}, c, F_0, \cdots, G_0, \cdots)$. Since $\mathfrak{B}$ is a model of Δ, $\mathfrak{B} \equiv \mathfrak{A}$. We shall show that $\mathfrak{B}$ has the property described in part (ii) of the lemma.

Let us list the elements of B, say $B = \{b_0, b_1, \cdots, b_n, \cdots\}$. Define the function f on B into B by

$$f(b_n) = F_n(b_0, b_1, \cdots, b_n).$$

This definition is unambiguous even if some b occurs more than once in the sequence $b_0, b_1, \cdots$, because of (3). We claim that f has the following three properties:

(4) Range of $f \neq B$.

(5) $U_{\mathfrak{B}} \subset$ range of f.

(6) For all formulas $\phi(x_0, \cdots, x_n)$ of L, if $b_0, \cdots, b_n$ satisfies ϕ in $\mathfrak{B}$ then so does $fb_0, \cdots, fb_n$.

Condition (4) is guaranteed by the sentences (1). Condition (5) is guaranteed by (2), because if $U(b_j)$ and $b_i = G_j(b_0, \cdots, b_j)$, then choosing $n \geqq i, j$ we have $f(b_i) = F_i(b_0, \cdots, b_i) = b_j$. Finally, condition (6) is guaranteed by (3).

Now let $\mathfrak{C}$ be the submodel of $\mathfrak{B}$ such that C is the range of f. It follows from (4) that $\mathfrak{C} \neq \mathfrak{B}$, and from (5) that $U_{\mathfrak{B}} \subset C$. From (6) we see that f is an isomorphism from $\mathfrak{B}$ to $\mathfrak{C}$, and it follows that $U_{\mathfrak{C}} = U_{\mathfrak{B}}$ and $\mathfrak{B} \cong \mathfrak{C}$. It also follows from (6) that $\mathfrak{C} \prec \mathfrak{B}$, because if $fb_0, \cdots, fb_n$ satisfies ϕ in $\mathfrak{C}$ then $b_0, \cdots, b_n$ satisfies ϕ in $\mathfrak{B}$ and hence $fb_0, \cdots, fb_n$ satisfies ϕ in $\mathfrak{B}$. By the lemma, there is a two-cadinal model which is equivalent to $\mathfrak{A}$. Our proof is complete.

There are several ways in which we can modify the scheme (*) without affecting the proof of Theorem 1. This gives us some other slightly different sets of axioms for two-cardinal models. One possibility is to replace the scheme (*) by

$$\exists v_0 \forall x_0 \exists y_0 z_0 \cdots \forall x_n \exists y_n z_n$$

$$(**) \qquad \left[\bigwedge_{i=0}^{n} v_0 \neq y_i \;\&\; \bigwedge_{i,j=0}^{n} (U(x_j) \to (x_i = z_j \leftrightarrow y_i = x_j)) \right.$$
$$\left. \& \bigwedge_{j=0}^{m} (\phi_j(x_0, \cdots, x_n) \leftrightarrow \phi_j(y_0, \cdots, y_n)) \right].$$

Another scheme of axioms for two-cardinal models which will work with the same proof is:

$$(***)\qquad \exists v_0 \forall x_0 w_0 \exists y_0 z_0 \cdots \forall x_n w_n \exists y_n z_n \left[\bigwedge_{i=0}^{n} v_0 \neq y_i \,\&\, \left[\left(\neg \exists v_1 U(v_1) \vee \bigwedge_{i=1}^{n} U(w_i) \right) \rightarrow \bigwedge_{j=0}^{m} (\phi_j(x_0, \cdots, x_n, z_0, \cdots, z_n) \leftrightarrow \phi_j(y_0, \cdots, y_n, w_0, \cdots, w_n)) \right] \right].$$

Everything works out just as well if we define the notion of a two-cardinal model in the following slightly different way. Let the language L have two unary predicates U, V, in addition to $P_0, P_1, \cdots$. By a two-cardinal model we now mean a model $\mathfrak{A}$ for L such that $V_{\mathfrak{A}}$ is infinite and the power of $U_{\mathfrak{A}}$ is less than the power of $V_{\mathfrak{A}}$. Then we get a set of axioms for two-cardinal models simply by adding the extra term $V(v_0)$ to the conjunction inside the quantifiers in the scheme (*).

References

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