DIFERENTIABLE FUNCTIONS WITH BOUNDED NONEMPTY SUPPORT ON BANACH SPACES

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1. Introduction. In [3, p. 26], S. Lang has raised the question whether or not a Banach space admits Fréchet differentiable partitions of unity. G. Restrepo [4] has shown that a Banach space with separable dual admits $C^1$ partitions of unity. In this note results are given which imply that, if the density character of a Banach space is strictly less than the density character of its dual, then the space does not admit Fréchet differentiable partitions of unity. This is an announcement of the results; detailed proofs will appear later.

2. Preliminaries. Let $X$ be a Banach space with norm $\rho$; let $S_x = \{ x : x \in X & \rho(x) = 1 \}$. $X$ will be said to admit a norm $\bar{\rho}$, if $\bar{\rho}$ is equivalent to $\rho$. The density character of $X$, denoted $\text{dens } X$, is the greatest lower bound of the cardinal numbers of dense subsets of $X$, or equivalently, the least upper bound of the cardinal numbers of discrete subsets of $X$. For $x, u \in X$, let

$$ (\rho'x)(u) = \lim_{t \to 0^+} \frac{\rho(x + tu) - \rho(x)}{t} $$

(this limit always exists). The support of a real or vector valued function $f$ on $X$ is the closure of the set $\{ x : x \in X & f(x) \neq 0 \}$. Throughout this note differentiable will mean Fréchet differentiable.

3. Results. The main result is

**Theorem 1.** If $X$ admits a norm $\bar{\rho}$ such that $\bar{\rho}'$ is uniformly discontinuous (i.e., there exists $\epsilon > 0$ such that for all $x \in X$ and $\delta > 0$, there are $x_1, x_2 \in X$ and $u \in S_x$ such that $\bar{\rho}(x_1 - x) < \delta$, $\bar{\rho}(x_2 - x) < \delta$ and $| (\bar{\rho}'x_1)(u) - (\bar{\rho}'x_2)(u) | > \epsilon$), then there exists no differentiable real valued function on $X$ with bounded nonempty support.

The proof of Theorem 1 uses methods similar to those of J. Kurzweil [2], who essentially proved the above result for the spaces $l_1$ and $C[0, 1]$ and continuously differentiable functions. R. Bonic and
J. Frampton [1] removed the continuity condition on the differential from Kurzweil's hypothesis.

**Theorem 2.** If $\text{dens } X < \text{dens } X^*$, then $X$ admits a norm $\hat{p}$ such that $\hat{p}'$ is uniformly discontinuous.

4. **Remarks.** It should also be noted that $\text{dens } X < \text{dens } X^*$ implies that there are no differentiable functions with bounded nonempty support from $X$ into any Banach space $Y$. For, if $f: X \to Y$ is differentiable and has bounded nonempty support, then there exists $g \in Y^*$ such that $g \circ f$ has bounded nonempty support (which is certainly contained in the support of $f$). Further, if any subspace of $X$ satisfies the hypothesis of Theorem 1, then the result is still true. Moreover, if $X$ is the cartesian product of $Y$ and another space, and if $Y$ admits a norm $\bar{p}$ such that $\bar{p}'$ is uniformly discontinuous, then $X$ has the same property. Finally, it may be observed that $X$ can be chosen to be $Y \times Z$ where $\text{dens } Y < \text{dens } Y^* = \text{dens } Z = \text{dens } Z^*$, so that $\text{dens } X = \text{dens } X^*$. This shows that $\text{dens } X = \text{dens } X^*$ does not imply the existence of a real valued differentiable function on $X$ with bounded nonempty support.

**Bibliography**


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