ON THE OSCILLATIONS AND LEBESGUE CLASSES
OF A FUNCTION AND ITS POTENTIALS

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Suppose \( f \in L^r(R) \), \( r \geq 1 \), \( R \) a cube in \( E^n \). Then one knows from Sobolev’s theorems [5] that the potential

\[
(0.1) \quad P \to \int_R f(Q) \left| P - Q \right|^{-\alpha} dQ, \quad 0 < \alpha < 1,
\]

is in \( L^s(R) \), \( s^{-1} > \alpha - 1 + r^{-1} \), where \( \left| P - Q \right| \) denotes the Euclidean distance between \( P, Q \in E^n \).

In this note we demonstrate a certain converse proposition. For a non-negative function \( f \in L^r(R) \), \( r \geq 1 \), we assume the potential (0.1) to be in \( L^s(R) \), \( 0 \leq s^{-1} < \alpha - 1 + r^{-1} \) (\( s \) a positive real number or \( \infty \)), and in addition make an assumption on the “oscillations” of \( f \) (cf. §1). Then we can conclude that \( f \) is summable to powers exceeding \( r \).

We express the so-called “oscillatory” conditions and present the main theorem, Theorem A, in the next section. The proof of the theorem is direct and simple. In §2 we state a parallel theorem, Theorem B, wherein the assumption on the potential is replaced by the hypothesis that the function is in some “Morrey class” (cf. Morrey [3]; or also Campanato [1]). Theorem B is described perhaps more accurately as a corollary to the proof of Theorem A. In the last section, §3, we show how these results can be indirectly deduced. Therein we use a lemma from a paper by Semenov [4] which relates “Marcinkiewicz classes” (cf. e.g., Zygmund [6]) with “Lorentz” spaces. The conclusion follows then from the inclusion relations between Lorentz spaces and Lebesgue spaces (cf. Lorentz [2]).

1. The principal result. Let \( f \) be a non-negative function summable over \( R \), a cube in \( E^n \). For \( S \) any measurable set in \( E^n \) we indicate its (Lebesgue) measure by \( \text{meas } S \). Set

\[
(1.1) \quad E(x) = \{ P : P \in R, \ f(P) > x \}.
\]

CONDITION I. For some \( a > 0 \), \( 0 \leq \lambda \leq 1 \) (\( a \) may depend on \( \lambda \))

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\begin{align}
(1.2) \quad xe^{E(x)1+\lambda} \leq \sup \int_C f(Q) \, dQ, \quad x > a,
\end{align}

where the supremum is taken over all parallel subcubes $C \subseteq \mathbb{R}$ with volume $meas \, E(x)$. Denote by $\lambda$ the infimum of the set of numbers $\lambda$ for which (1.2) holds.

If one considers the inequality (1.2) for some fixed $x$, then it can be interpreted as a condition on the dispersion of the set of points where the function assumes large values (exceeding $x$). It is for this reason that we refer to the foregoing as a condition on the oscillations of a function (and, similarly, for the alternate conditions presented later in this section).

\textbf{Remark.} One property of the quantity $\lambda$ is that its reciprocal measures what one might describe as the upper bound (with respect to the exponent) of the Lebesgue classes of $f$. That is, $f$ is at best in $L^{1/\lambda}$. This observation, however, is seemingly not very interesting. For consider the situation on the line: $n = 1$, $R$ and $C$ intervals. It is clear that for a monotone function $\lambda = 0$. Whereas there are monotone functions in $L^p$ and not in $L^{p+\epsilon}$ for any $p$ and $\epsilon > 0$.

\textbf{Theorem A.} Suppose $f \in L^r(R), \ r \geq 1,$ and the potential (0.1) is in $L^s(R)$ where $0 \leq s^{-1} < \alpha - 1 + r^{-1}$. If $\lambda < \alpha - 1 + r^{-1} - s^{-1}$, then $f \in L^p(R)$ for $p < (1 - \alpha + s^{-1} + \lambda)^{-1}$.

\textbf{Proof.} It follows simply using Hölder's inequality that

\begin{align}
(1) \quad \int_C \int_R f(Q) \, |P - Q|^{-an} \, dQ \, dP \leq \text{(Constant)} \, (\text{meas} \, C)^{1-s^{-1}}.
\end{align}

Now the left side in (1) dominates the quantity

\begin{align}
(2) \quad \text{(dia} \, C)^{-an} \, (\text{meas} \, C) \int_C f(Q) \, dQ.
\end{align}

From (1), (2) and the fact that $(\text{dia} \, C)^{-n} = n^{n/2} (\text{meas} \, C)$ we find that

\begin{align}
(1.3) \quad \int_C f(Q) \, dQ \leq \text{(Constant)} \, (\text{meas} \, C)^{\alpha - s^{-1}}.
\end{align}

Set $\lambda = \lambda + \epsilon$ where $\epsilon > 0$ is any number satisfying the inequality $\lambda + \epsilon < \alpha - 1 + r^{-1} - s^{-1}$. Then on combining (1.2) and (1.3), for sufficiently large $x$, it follows that

\[\lambda = \lambda + \epsilon < \alpha - 1 + r^{-1} - s^{-1}.\]

\[\int_C f(Q) \, dQ \leq \text{(Constant)} \, (\text{meas} \, C)^{\alpha - s^{-1}}.\]
OSCILLATIONS AND LEBESGUE CLASSES OF A FUNCTION

\[ x (\text{meas } E(x))^{1+\lambda} \leq (\text{Constant}) (\text{meas } E(x))^{a^{-1}}; \]

or

\[ \text{meas } E(x) \leq (\text{Constant}) \left( \frac{1}{x} \right)^{1/(1+\lambda-a^{-1})}. \tag{3} \]

The desired conclusion results from (3), the boundedness of \( \text{meas } E(x) \), and the fact that

\[ \int_R [f(Q)]^p dQ = \rho \int_0^\infty (\text{meas } E(x)) x^{p-1} dx. \]

We shall present now two other conditions that can be used instead of Condition I. The three conditions are ordered according to increasing relative strengths.

**CONDITION II.** Let \( f^* = f^*(t) \), \( 0 < t < \text{meas } R \), be a decreasing function equi-measurable with \( f \). Set

\[ \log \left( \sup \int_C f(P) dP / \int_0^t f^*(t) dt \right) \]

\[ \mu = \limsup_{\varepsilon \to 0} \log \varepsilon \]

where the supremum in the numerator is to be taken over all parallel subcubes \( C \subset R \) with volume \( \varepsilon \).

**CONDITION III.** Suppose \( \text{meas } E(x) > 0 \), \( x > 0 \). Set

\[ 1 + \nu = \limsup \frac{\log[\sup \text{meas } (E(x) \cap C)]}{\log(\text{meas } E(x))} \]

where \( \sup \text{meas } (E(x) \cap C) \) is taken over all parallel subcubes \( C \subset R \) with volume \( \text{meas } E(x) \).

**REMARK.** The theorem then holds with \( \mu \) or \( \nu \) in place of \( \lambda \).

We observe further that a more local type theory could be developed based on local conditions similar to the above. For example, consider Condition II: For \( P \) fixed in \( R \) formulate (1.4) for a cube with center \( P \) and contained in \( R \). Then shrink the cube down to \( P \).

2. **A parallel theorem.** Let \( S \) be a bounded open set in \( E^n \) of diameter \( \rho_0 \). Denote by \( B(P, \rho) \) the ball with center \( P \) and radius \( \rho \). Let \( q \) and \( \delta \) be real numbers where \( q \geq 1 \) and \( 0 \leq \delta \leq n \). A function \( f \) is said to be in the Morrey class \( L^{(q,\delta)}(S) \) if there exists a constant \( K \) such that

\[ \int_{B(P,\rho) \cap S} |f(Q)|^q dQ \leq K \rho^\delta \]
for all \( P \subseteq S \) and \( 0 \leq \rho \leq \rho_0 \).

We apply this definition in a slightly modified form. Here \( S \) is \( R \), a cube in \( E^n \). We take cubes \( C = C(P, \rho) \) of diameter \( \rho \) centered at points \( P \) of \( R \) instead of balls. Then we replace relation (2.1) by the equivalent relation

\[
\left( \int_{C \cap R} |f(Q)|^\beta dQ \right)^{1/\beta} \leq K(\text{meas } C) \rho
\]

where \( 0 \leq \beta \leq g^{-1} \). We denote the corresponding class now by \( L^{(q, \beta)}(R) \).

**Theorem B.** Let \( f \) be a non-negative function in \( L^{(q, \beta)}(R) \). If \( \lambda < \beta \) then \( f \in L^p(R) \) where \( p < (g^{-1} - \beta + \lambda)^{-1} \).

The proof parallels that of Theorem A. Instead of relation (1.3) we deduce in this case using Hölder's inequality and (2.2) that

\[
\int_{C} f(Q) \, dQ \leq (\text{Constant}) \ (\text{meas } C)^{1-a^{-1}+\beta}.
\]

The proof is completed then just as in the proof of Theorem A.

3. **An indirect proof.** A function \( f \) measurable on \( R \) is said to be in the Lorentz space \( M(\gamma) \), \( 0 \leq \gamma \leq 1 \), provided that

\[
||f||_{M(\gamma)} = \sup_{0 < t < \text{meas } R} \frac{\int_0^t f^\ast(t) \, dt}{\gamma} < \infty
\]

where \( f^\ast = f^\ast(t) \), \( 0 < t < \text{meas } R \), is a decreasing function equi-measurable with \( |f| \).

We shall say that a function \( f \) in \( R \) is in the Marcinkiewicz class \( \mathcal{M}(\gamma) \), \( 0 \leq \gamma \leq 1 \), if it satisfies the condition

\[
\sup_{0 < x < \infty} x(\text{meas } E(x))^{1-\gamma} < \infty
\]

where \( \text{meas } E(x) \) is the distribution function of \( |f| \).

The following lemma which relates Lorentz spaces and Marcinkiewicz classes appears in [4].

**Lemma.** The Marcinkiewicz class \( \mathcal{M}(\gamma) \) coincides with the space \( M(\gamma) \). In addition

\[
\sup_x x(\text{meas } E(x))^{1-\gamma} \leq ||f||_{M(\gamma)} \leq \gamma^{-1} \sup_x x(\text{meas } E(x))^{1-\gamma}.
\]
Now consider again the proof of Theorem A. We deduce from (3.2), using relation (3) in the proof, that \( f \in M(\alpha - \lambda - s^{-1}) \). Then on applying the Lemma it follows that \( f \in M(\alpha - \lambda - s^{-1}) \). The desired conclusion is derived finally from the inclusion relation \( M(\gamma) \subset L^{(1-\gamma)^{-1}} \), \( \gamma' < \gamma \).

REFERENCES

1. S. Campanato, Proprietà di inclusione per spazi di Morrey, Ricerche Mat. 12 (1963), 67–86.

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