ON THE OSCILLATIONS AND LEBESGUE CLASSES OF A FUNCTION AND ITS POTENTIALS[1]

BY R. K. JUBERG

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Suppose $f\in L^r(R)$, $r\geqq 1$, R a cube in E^n. Then one knows from Sobolev's theorems [5] that the potential

$$(0.1)\qquad P\to\int_R f(Q)\,|P-Q|^{-\alpha n}\,dQ,\qquad 0<\alpha<1,$$

is in $L^\sigma(R)$, $\sigma^{-1}>\alpha-1+r^{-1}$, where $|P-Q|$ denotes the Euclidean distance between P, $Q\in E^n$.

In this note we demonstrate a certain converse proposition. For a non-negative function $f\in L^r(R)$, $r\geqq 1$, we assume the potential (0.1) to be in $L^s(R)$, $0\leqq s^{-1}<\alpha-1+r^{-1}$ (s a positive real number or ∞), and in addition make an assumption on the "oscillations" of f (cf. §1). Then we can conclude that f is summable to powers exceeding r.

We express the so-called "oscillatory" conditions and present the main theorem, Theorem A, in the next section. The proof of the theorem is direct and simple. In §2 we state a parallel theorem, Theorem B, wherein the assumption on the potential is replaced by the hypothesis that the function is in some "Morrey class" (cf. Morrey [3]; or also Campanato [1]). Theorem B is described perhaps more accurately as a corollary to the proof of Theorem A. In the last section, §3, we show how these results can be indirectly deduced. Therein we use a lemma from a paper by Semenov [4] which relates "Marcinkiewicz classes" (cf. e.g., Zygmund [6]) with "Lorentz" spaces. The conclusion follows then from the inclusion relations between Lorentz spaces and Lebesgue spaces (cf. Lorentz [2]).

1. **The principal result.** Let f be a non-negative function summable over R, a cube in E^n. For S any measurable set in E^n we indicate its (Lebesgue) measure by meas S. Set

$$(1.1)\qquad E(x)=\{P\colon P\in R,\ \ f(P)>x\}.$$

CONDITION I. For some $a>0$, $0\leqq\lambda\leqq 1$ (a may depend on λ)

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$$(1.2) \qquad x(\text{meas } E(x))^{1+\lambda} \leqq \sup \int_C f(Q)\, dQ, \qquad x > a,$$

where the supremum is taken over all parallel subcubes $C \subset R$ with volume meas $E(x)$. Denote by $\bar{\lambda}$ the infimum of the set of numbers λ for which (1.2) holds.

If one considers the inequality (1.2) for some fixed x, then it can be interpreted as a condition on the dispersion of the set of points where the function assumes large values (exceeding x). It is for this reason that we refer to the foregoing as a condition on the oscillations of a function (and, similarly, for the alternate conditions presented later in this section).

REMARK. One property of the quantity $\bar{\lambda}$ is that its reciprocal measures what one might describe as the upper bound (with respect to the exponent) of the Lebesgue classes of f. That is, f is at best in $L^{1/\bar{\lambda}}$. This observation, however, is seemingly not very interesting. For consider the situation on the line: $n=1$, R and C intervals. It is clear that for a monotone function $\bar{\lambda}=0$. Whereas there are monotone functions in L^p and not in $L^{p+\epsilon}$ for any p and $\epsilon > 0$.

THEOREM A.[2] *Suppose* $f \in L^r(R)$, $r \geqq 1$, *and the potential* (0.1) *is in* $L^s(R)$ *where* $0 \leqq s^{-1} < \alpha - 1 + r^{-1}$. *If* $\bar{\lambda} < \alpha - 1 + r^{-1} - s^{-1}$, *then* $f \in L^p(R)$ *for* $p < (1 - \alpha + s^{-1} + \bar{\lambda})^{-1}$.

PROOF. It follows simply using Hölder's inequality that

$$(1) \qquad \int_C \int_R f(Q) \,|\, P - Q\,|^{-\alpha n}\, dQ\, dP \leqq (\text{Constant})\, (\text{meas } C)^{1-s^{-1}}.$$

Now the left side in (1) dominates the quantity

$$(2) \qquad (\text{dia } C)^{-\alpha n}\, (\text{meas } C) \int_C f(Q)\, dQ.$$

From (1), (2) and the fact that $(\text{dia } C)^n = n^{n/2}(\text{meas } C)$ we find that

$$(1.3) \qquad \int_C f(Q)\, dQ \leqq (\text{Constant})\, (\text{meas } C)^{\alpha - s^{-1}}.$$

Set $\lambda = \bar{\lambda} + \epsilon$ where $\epsilon > 0$ is any number satisfying the inequality $\bar{\lambda} + \epsilon < \alpha - 1 + r^{-1} - s^{-1}$. Then on combining (1.2) and (1.3), for sufficiently large x, it follows that

[2] This theorem was presented in preliminary form at the 69th Summer Meeting, American Mathematical Society, Amherst, Massachusetts, August 25–28, 1964. Notices Amer. Math. Soc. 11 (1964), 574.

$$x(\text{meas } E(x))^{1+\lambda} \leqq (\text{Constant})\, (\text{meas } E(x))^{\alpha-s^{-1}};$$

or

$$\text{meas } E(x) \leqq (\text{Constant}) \left(\frac{1}{x}\right)^{1/(1+\lambda-\alpha+s^{-1})} . \tag{3}$$

The desired conclusion results from (3), the boundedness of meas $E(x)$, and the fact that

$$\int_R [f(Q)]^p \, dQ = p \int_0^\infty (\text{meas } E(x)) x^{p-1} \, dx.$$

We shall present now two other conditions that can be used instead of Condition I. The three conditions are ordered according to increasing relative strengths.

CONDITION II. Let $f^* = f^*(t)$, $0 < t < \text{meas } R$, be a decreasing function equi-measurable with f. Set

$$\mu = \limsup_{z \to 0} \frac{\log\left(\sup \int_C f(P)dP \Big/ \int_0^z f^*(t)dt\right)}{\log z} \tag{1.4}$$

where the supremum in the numerator is to be taken over all parallel subcubes $C \subset R$ with volume z.

CONDITION III. Suppose meas $E(x) > 0$, $x > 0$. Set

$$1 + \nu = \limsup_{x \to \infty} \frac{\log[\sup \text{meas}(E(x) \cap C)]}{\log(\text{meas } E(x))} \tag{1.5}$$

where sup meas $(E(x) \cap C)$ is taken over all parallel subcubes $C \subset R$ with volume meas $E(x)$.

REMARK. The theorem then holds with μ or ν in place of λ.

We observe further that a more local type theory could be developed based on local conditions similar to the above. For example, consider Condition II: For P fixed in R formulate (1.4) for a cube with center P and contained in R. Then shrink the cube down to P.

2. **A parallel theorem.** Let S be a bounded open set in E^n of diameter ρ_0. Denote by $B(P, \rho)$ the ball with center P and radius ρ. Let q and δ be real numbers where $q \geqq 1$ and $0 \leqq \delta \leqq n$. A function f is said to be in the *Morrey* class $L^{(q,\delta)}(S)$ if there exists a constant K such that

$$\int_{B(P,\rho)\cap S} |f(Q)|^q \, dQ \leqq K\rho^\delta \tag{2.1}$$

for all $P \in S$ and $0 \leqq \rho \leqq \rho_0$.

We apply this definition in a slightly modified form. Here S is R, a cube in E^n. We take cubes $C = C(P, \rho)$ of diameter ρ centered at points P of R instead of balls. Then we replace relation (2.1) by the equivalent relation

$$\left(\int_{C \cap R} |f(Q)|^q \, dQ \right)^{1/q} \leqq K(\text{meas } C)^\beta \tag{2.2}$$

where $0 \leqq \beta \leqq q^{-1}$. We denote the corresponding class now by $L^{(q,\beta)}(R)$.

THEOREM B. *Let f be a non-negative function in $L^{(q,\beta)}(R)$. If $\bar{\lambda} < \beta$ then $f \in L^p(R)$ where $p < (q^{-1} - \beta + \bar{\lambda})^{-1}$.*

The proof parallels that of Theorem A. Instead of relation (1.3) we deduce in this case using Hölder's inequality and (2.2) that

$$\int_C f(Q) \, dQ \leqq (\text{Constant}) \, (\text{meas } C)^{1 - q^{-1} + \beta}.$$

The proof is completed then just as in the proof of Theorem A.

3. **An indirect proof.** A function f measurable on R is said to be in the *Lorentz* space $M(\gamma)$, $0 \leqq \gamma \leqq 1$, provided that

$$\|f\|_{M(\gamma)} = \sup_{0 < s < \text{meas } R} \frac{\int_0^s f^*(t) dt}{s^\gamma} < \infty \tag{3.1}$$

where $f^* = f^*(t)$, $0 < t < \text{meas } R$, is a decreasing function equi-measurable with $|f|$.

We shall say that a function f in R is in the *Marcinkiewicz* class $\mathfrak{M}(\gamma)$, $0 \leqq \gamma \leqq 1$, if it satisfies the condition

$$\sup_{0 < x < \infty} x(\text{meas } E(x))^{1-\gamma} < \infty \tag{3.2}$$

where meas $E(x)$ is the distribution function of $|f|$.

The following lemma which relates Lorentz spaces and Marcinkiewicz classes appears in [4].

LEMMA. *The Marcinkiewicz class $\mathfrak{M}(\gamma)$ coincides with the space $M(\gamma)$. In addition*

$$\sup_x x(\text{meas } E(x))^{1-\gamma} \leqq \|f\|_{M(\gamma)} \leqq \gamma^{-1} \sup_x x(\text{meas } E(x))^{1-\gamma}.$$

Now consider again the proof of Theorem A. We deduce from (3.2), using relation (3) in the proof, that $f\in\mathfrak{M}(\alpha-\lambda-s^{-1})$. Then on applying the Lemma it follows that $f\in M(\alpha-\lambda-s^{-1})$. The desired conclusion is derived finally from the inclusion relation $M(\gamma)\subset L^{(1-\gamma')^{-1}}$, $\gamma'<\gamma$.

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University of Minnesota