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EMBEDDING 1-CONNECTED MANIFOLDS

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In this note we announce some results on existence and isotopy of embeddings. In the first section we consider the case of codimension 1 and study the problem of smooth or piecewise linear embedding of manifolds in manifolds of one dimension higher. The isotopy result may be considered as a generalization of the Schönflies theorem. In the second section we attack the problem of smooth embedding for codimension $k>2$ and here our results generalize results of Levine [**6**], who treated the problem of knotted spheres. For codimensions $k>2$ our results reduce problems of embedding 1-connected manifolds of dimension $\geqq 5$ in 2-connected manifolds, to problems in homotopy theory. Codimension 2 appears more difficult as usual, and the techniques used here do not seem to work for that case. In the third section, for the metastable range we obtain new existence and classification theorems for embeddings (rather than embedded submanifolds as in §2). Levine [**5**] proved analogous results for embedding in S^{n+k} but there is no obvious relation between his theorems and ours, even when the bigger manifold is taken to be S^{n+k}.

We state the existence and isotopy theorems for closed manifolds, but analogous results for manifolds with boundary hold, for boundary embedded in boundary.

All embeddings will be smooth unless explicitly stated to the contrary.

The full exposition will appear later.

1. **Codimension** 1. Let $f\colon M\to X$, $g\colon M\to Y$ be maps and define the *double mapping cylinder* of f and g to be the space $Z=X\cup M\times[-1,1]\cup Y$ with the identifications $(x,-1)=f(x)\in X$, $(x,1)=g(x)\in Y$ for all $x\in M$. We write $Z=X\cup_f M\times I\cup_g Y$. By replacing X and Y by mapping cylinders of f and g, we get $Z=X\cup Y$, $M=X\cap Y$.

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A space M is said to satisfy Poincaré duality in dimension n if $H_nM=Z$, with generator g, and $\cap g: H^i(M)\to H_{n-i}(M)$ is an isomorphism for every i.

If (X_1, X_2, X_3, X_4) and (Y_1, Y_2, Y_3, Y_4) are triads, $X_1=X_2\cup X_3$, $X_4=X_2\cap X_3$, $Y_1=Y_2\cup Y_3$, $Y_4=Y_2\cap Y_3$, a map $h: X_1\to Y_1$ is said to be a homotopy equivalence of triads if $h(X_i)\subset Y_i$ and $h|X_i$ is a homotopy equivalence of X_i with Y_i, $i=1, 2, 3, 4$.

THEOREM 1.1. *Let M be a 1-connected finite polyhedron satisfying Poincaré duality in dimension $n\geqq 5$. Let X, Y be spaces with $H_nX=H_nY=0$, $f: M\to X$, $g: M\to Y$ maps such that $f_*: H_2M\to H_2X$, $g_*: H_2M\to H_2Y$ are onto and let $Z=X\cup_f M\times I\cup_g Y$. Finally let W^{n+1} be a closed 1-connected smooth (piecewise linear) manifold and $h: W\to Z$ a homotopy equivalence. Then there is a smooth (p.l.) n-submanifold $N^n\subset W^{n+1}$ such that $W=A\cup B$, A, B, $(n+1)$-manifolds with boundary $\partial A=\partial B=N=A\cap B$, and such that h is homotopic to a homotopy equivalence of triads, (W, A, B, N) with (Z, X, Y, M). Further, such N, A, B, are unique up to a pseudo-isotopy*[2] *of W. (Compare* [2] *and* [8].)

Note that the condition on H_2 is automatically satisfied if W is 2-connected.

A homotopy equivalence of smooth manifolds $f: M_1\to M_2$ is called a tangential equivalence if $f^*(\tau_{M_1}\oplus\epsilon^1)=\tau_{M_2}\oplus\epsilon^1$ where τ_{M_i} is the tangent bundle of M_i, $\epsilon^q=$ trivial q plane bundle. An analogous definition makes sense in the case of p.l. manifolds, using the tangent microbundles (see [7]).

THEOREM 1.2. *With the hypothesis of* (1.1), *if in addition M is a smooth (p.l.) n-manifold and if $(k|M)^*(\tau_W)=\tau_M\oplus\epsilon^1$ as vector bundles (as p.l. microbundles), where k is a homotopy inverse of h, then M is tangentially equivalent to N.*

Let ν_W be the normal bundle (microbundle) to W for an embedding of $W\subset S^{n+k}$, k large. Then the homotopy class $c_W\in\pi_{n+k}(T\nu_W)$ of the collapsing map $S^{n+k}\to T(\nu_W)=$ the Thom complex of ν_W will be called a "normal invariant of W," (similarly for M). If ξ is a bundle over Z, $\xi|M\times(\frac{1}{4},\frac{3}{4})$ has total space $E(\xi|M\times(\frac{1}{4},\frac{3}{4}))=E(\xi|M\times\frac{1}{2})\times(\frac{1}{4},\frac{3}{4})=E((\xi|M\times\frac{1}{2})\oplus\epsilon^1)$. Then there is a natural collapsing map $\eta: T(\xi)\to T((\xi|M)\oplus\epsilon^1)$. If $(k|M)^*(\tau_W)=\tau_M\oplus\epsilon^1$ it follows that $(k|M)^*\nu_W$ is stably equivalent to ν_M so that if we set $\xi=k^*(\nu_W)$, $\xi|M=(k|M)^*(\nu_W)$,

[2] Embedded smooth submanifolds M_1, $M_2\subset V$ are pseudo-isotopic if there exists a diffeomorphism F of $V\times I$ with $F(x, 0)=(x, 0)$, $F(x, 1)=(f(x), 1)$ and $fM_1=M_2$.

and $(\xi|M)\oplus\epsilon^1$ is equivalent to ν_M for some embedding of $M\subset S^{n+k}$, k large. (In the piecewise linear case we must add trivial bundles ϵ^q for some q to both sides.) Then considering η: $T(\nu_W)\to T(\nu_M)$ we get:

THEOREM 1.3. *With hypotheses as in* (1.1) *and* (1.2), *if in addition* $\eta_*(c_W)=c_M$, *some normal invariant for* M, *then* M *is diffeomorphic to* $N\#\Sigma$, *where* Σ *is a homotopy n-sphere which bounds a parallelizable manifold. In particular, if n is even,* M *is diffeomorphic to* N. (*In the piecewise linear case,* M *is p.l. equivalent to* N *for all n.*)

2. **Codimension** $k>2$. Given a manifold M^n, we first consider the problem of embedding it in a manifold W^{n+k} from the point of view of finding a k-plane bundle ξ over M which we would like to be the normal bundle, and applying the results of §1 to embed the sphere bundle E_0 up to homotopy or tangential equivalence in W, with one complement homotopy equivalent to E, the disk bundle of ξ. Then we show that one complement is a disk bundle. Our results may be considered generalizations of [6].

THEOREM 2.1. *Let* M^n, W^{n+k} *be closed,* 1-*connected, smooth manifolds,* $n\geq 5$, $k>2$, ξ^k *a k-plane bundle over* M, E, E_0, *total spaces of the disk bundle and sphere bundle respectively associated to* ξ. *Let* Y *be a space with* $H_{n+k-1}Y=0$, $f\colon E_0\to Y$ *a map such that* $f_*\colon H_2E_0\to H_2Y$ *is onto, and let* $Z=E\cup_f Y$. *Let* $h\colon Z\to W$ *be a homotopy equivalence such that* $h^*\tau_W|E=\tau_E$. *If* $n\not\equiv 2$ mod 4, *then there is a submanifold* N^n *of* W^{n+k}, *and a map* $h'\sim h^{-1}$, *such that* $h'(N)\subset M$, h' *is a tangential equivalence of* N *with* M, $h'^*(\xi)=$ *normal bundle of* N *in* W, *and* $h'|W-N$ *is a homotopy equivalence with* $Z-M\cong Y$.

In (2.1), since $(h^*\tau_W)|E=\tau_E$, it follows that if ν_W^q is the normal bundle to W^{n+k} in S^{n+k+q} for q large, then $(h^*\nu_W^q)|E=\nu_E^q$ and hence $((h|M)^*\nu_W^q)\oplus\xi^k=\nu_M^{k+q}$. Then since the total space of ν_M is an open subspace of the total space of $h^*(\nu_W)$, and h is a homotopy equivalence, there is a natural collapsing map η: $T(\nu_W)\to T(\nu_M)$.

THEOREM 2.2. *With the hypothesis of* (2.1), *but for any* $n\geq 5$, $k>2$, *if in addition* $\eta_*(c_W)=c_M$, *where* $c_W\in\pi_{n+k+q}(T(\nu_W))$ *is a normal invariant for* W *and* $c_M\in\pi_{n+k+q}(T(\nu_M))$ *is a normal invariant for* M, *then* M *can be embedded in* W *with normal bundle* ξ, *and complement* $W-M$ *homotopy equivalent to* Y, as in (2.1).

The set of pseudo-isotopy classes of pairs (S^{n+k}, Σ^n), where Σ^n is a homotopy n-sphere $\subset S^{n+k}$, forms a group $\theta^{n+k,n}$ under connected sum if $k>2$ (see [4] and [6]). Then $\theta^{n+k,n}$ acts by taking the connected sum on the set of pseudo-isotopy classes of (W^{n+k}, M_α^n) where M_α^n

is tangentially equivalent to M^n. Let $J \subset \theta^{n+k,n}$, $k>2$, $n=2q-1$, be the subgroup generated by $(S^{n+k}, \Sigma^n)=\partial(D^{n+k+1}, U)$, where U is a framed $(q-1)$-connected $2q$-manifold in D^{n+k+1} of index 8 if q is even, or Arf invariant 1 if q is odd, so that Σ^n generates $\theta^n(\partial\pi)$, (see [6]). Define $J=0$ if n is even. Note that $J=\partial_3 P_{n+1}$ in notation of [6].

For a closed manifold M^n let $J(M)\subset J$ be the subgroup of J of pairs (S^{n+k}, Σ^n) such that $\Sigma \# M$ is diffeomorphic to M. Since $\theta^n(\partial\pi)$ is cyclic of finite order it follows that $J(M)$ is cyclic of finite index in J, with index $[J: J(M)]$ dividing the order of $\theta^n(\partial\pi)$. In particular $J(M)$ always contains J' = subgroup of J of index (order of $\theta^n(\partial\pi)$) consisting of pairs (S^{n+k}, Σ^n), where Σ^n is diffeomorphic to S^n, and in some cases is equal to that subgroup, for example for homotopy spheres, or certain π-manifolds (see [1]). If k is in the metastable range $k>(n+3)/2$, then $\theta^{n+k,n}$ is isomorphic to θ^n and the subgroup J' is trivial, although $J(M)$ may still be nontrivial. For k below the metastable range J' is infinite cyclic for $n=4q-1$ (see [6] and [4]).

Let (W^{n+k}, M_1^n) and (W^{n+k}, M_2^n) be two embeddings of smooth manifolds in W^{n+k}, with normal bundles ξ_1 and ξ_2 respectively. Let $E_0(\xi_i)$ = the $(k-1)$-sphere bundle associates to ξ_i, $E(\xi_i)$ the k-disk bundle, $i=1, 2$, and let $u_i: E_0(\xi_i)\to W-M_i$ be the inclusion of the boundary of a tubular neighborhood of M_i into the complement of M_i.

THEOREM 2.3. *Let M and W be closed, 1-connected, $n\geqq 5$, $k>2$, and suppose $u_{i*}: H_2(E_0(\xi_i))\to H_2(W-M_i)$ is onto, $i=1, 2$. Then (W, M_1) is pseudo-isotopic to (W, M_2) modulo the action of J if and only if there exists a bundle map $f: \xi_1\to\xi_2$, covering a homotopy equivalence $e: M_1\to M_2$ and a homotopy equivalence $g: W-M_1\to W-M_2$ such that*

(1) the diagram

$$\begin{array}{ccc} E_0(\xi_1) & \xrightarrow{f_0} & E_0(\xi_2) \\ \downarrow u_1 & & \downarrow u_2 \\ W-M_1 & \xrightarrow{g} & W-M_2 \end{array}$$

commutes up to homotopy and

(2) the map $h: W\to W$ induced by e, f and g is homotopic to the identity of W, where to define h we consider W as the double mapping cylinder of the maps $v_i: E_0(\xi_i)\to E(\xi_i)$ (the inclusion of the space bundle into the disk bundle) and $u_i: E_0\to W-M_i$.

If M_1 is diffeomorphic to M_2 replace J by $J(M_1)$.

For $W=S^{n+k}$ we have the following corollaries of (2.1)–(2.3).

COROLLARY 2.4. *Let M^n be a 1-connected closed smooth oriented manifold $n \geqq 5$ and ξ^k be a k-plane bundle over M, $k \geqq 2$. Suppose $\alpha \in \pi_{n+k}(T(\xi))$ is such that $h(\alpha) = \Phi([M])$, where h = Hurewicz homomorphism, Φ = Thom isomorphism, $[M]$ = the orientation class of $M \in H_n(M)$.*

(1) *If n is odd then there is a homotopy equivalent manifold $M' \subset S^{n+k+1}$ with normal bundle $\xi \oplus \epsilon^1$ and normal invariant $\Sigma(\alpha)$.*

(2) *If $n = 4q$ and $L_q(\bar{p}_1(\xi), \cdots, \bar{p}_q(\xi))$ = index of M, the same conclusion holds, where $\bar{p}_i(\xi)$ are the dual Pontrjagin classes of ξ, and L_q is the Hirzebruch polynomial.*

(3) *If $\tau_M \oplus \xi$ is trivial, and $n \not\equiv 2$* mod 4, *then M is tangentially equivalent to $M' \subset S^{n+k+1}$ with normal bundle $\xi \oplus \epsilon^1$ and normal invariant $\Sigma(\alpha)$.*

(4) *If $\xi \oplus \epsilon^s$ is equivalent to the stable normal bundle ν^{k+s} to $M \subset S^{n+k+s}$, s large (which is equivalent to $\tau_M \oplus \xi$ being trivial) and if in addition $\Sigma^s(\alpha) \in \pi_{n+k+s}(T(\nu))$ is a normal invariant for M then $M \subset S^{n+k+1}$ with normal bundle $\xi \oplus \epsilon^1$ and normal invariant $\Sigma(\alpha)$ (where we have used the bundle equivalence to identify $\xi \oplus \epsilon^s$ with ν).*

Note that $E_0(\xi \oplus \epsilon^1)/$(canonical cross section) is homeomorphic to $T(\xi)$. The condition on α implies that $T(\xi) \cong S^{n+k} \vee T(\xi_0)$, where $\xi_0 = \xi | M -$(point). Hence one may take $Y = T(\xi_0)$ and apply (2.1)–(2.2) to the bundle $\xi \oplus \epsilon_1$.

Let (S^{n+k}, M_i^n), $i = 1, 2$, be embeddings of 1-connected closed smooth manifolds M_1 and M_2, with oriented normal bundles ν_i and normal invariants $\alpha_i \in \pi_{n+k}(T(\nu_i))$, $i = 1, 2$. Applying (2.3) we get:

COROLLARY 2.5. *Let $b: \nu_1 \to \nu_2$ be a bundle map covering a homotopy equivalence of degree $+1$ of M_1 with M_2, and let $T(b): T(\nu_1) \to T(\nu_2)$ be the induced map of Thom complexes. Then the embedded manifolds M_1 and M_2 are pseudo-isotopic modulo J one dimension higher, i.e. in S^{n+k+1}. (If M_1 is diffeomorphic to M_2, replace J by $J(M_1)$.)*

3. **The metastable range.** By a $(k-1)$-spherical fibre space $\xi: E_0 \to^{\pi} M$, we mean that π is a fibre map in the sense of Serre and the fibre $\pi^{-1}(x)$ is homotopy equivalent to S^{k-1} (any $x \in M$).

THEOREM 3.1. *Let M^n be a smooth, closed, 1-connected manifold, $\xi: E_0 \to^{\pi} M$ a $(k-1)$-spherical fibre space over M, $k \geqq (n+3)/2$, $n+k \geqq 6$. Let $f: E_0 \to Y$ be a map, such that $f_*: H_2 E_0 \to H_2 Y$ is onto, $Z = M \cup_{\pi} E_0 \times I \cup_f Y$ the double mapping cylinder, and suppose Z is homotopy equivalent to a smooth 1-connected closed manifold W^{n+k}. Then the map $M \to W$ obtained from the composition $M \to Z \to W$ is homotopic to an embedding of M^n in W^{n+k} with normal bundle fibre homotopy equivalent to ξ. (Compare with (2.2).)*

For using (1.1) we get an embedded $(n+k)$-submanifold $A \subset W$ and $(A, \partial A)$ homotopy equivalent to (M, E_0). Then the homotopy equivalence $M \rightarrow A$ is homotopic to an embedding of M in A by a theorem of Haefliger [3]. It follows easily that the normal bundle is fibre homotopy equivalent to E_0.

THEOREM 3.2. *Let M^n be a smooth, closed, 1-connected manifold, W^{n+k} smooth closed 1-connected manifold, $k>(n+3)/2$, $n+k \geqq 5$, and let f_1, f_2 be two embeddings of M^n in W^{n+k} such that f_{1*} and $f_{2*}: H_2M \rightarrow H_2W$ are onto, and let ν_1 and ν_2 be the normal bundles of f_1 and f_2 respectively. Then f_1 is isotopic to f_2 if and only if there is a fibre homotopy equivalence $g: E_0(\nu_1) \rightarrow E_0(\nu_2)$ and a homotopy equivalence $h: W-f_1M \rightarrow W-f_2M$ such that the diagram*

$$\begin{array}{ccc} E_0(\nu_1) & \rightarrow & E_0(\nu_2) \\ \downarrow & & \downarrow \\ W - f_1M & \rightarrow & W - f_2M \end{array}$$

commutes, and such that the induced map of $W \rightarrow W$ is homotopic to the identity. (*Compare with* (2.3).)

These results seem related to those of Levine [5]. Levine's theorems do not seem to follow from ours, but we have the following corollary analogous to Levine's situation.

If $\xi: E_0 \rightarrow^{\pi} M$ is a $(k-2)$-spherical fibre space, let the Thom complex $T(\xi) = M \cup_{\pi} c(E_0)$, where $c(E_0)$ is the cone on E_0.

COROLLARY 3.3. *Let M^n be a closed smooth 1-connected manifold, and let $\xi: E_0 \rightarrow^{\pi} M$ be a $(k-2)$-spherical fibre space, $k \geqq (n+3)/2$, $n+k \geqq 6$ and let $\alpha \in \pi_{n+k-1}(T(\xi))$ be an element which goes onto the generator of $H_{n+k-1}(T\xi)$ under the Hurewicz homomorphism. Then there is an embedding of M in S^{n+k} with normal bundle fibre homotopy equivalent to $\xi \oplus \epsilon^1$, (the $(k-1)$-spherical fibre space obtained from Whitney sum with trivial S^0 bundle).* (*Compare with* (2.4).)

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