MULTILINEAR LEBESGUE-BOCHNER-STIELTJES INTEGRAL

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Communicated by A. Zygmund, November 22, 1965

In this paper we introduce an integral of the form $\int u(f_{ji}, d\mu_j)$ where u is a multilinear operator from the product of the Banach spaces Y_{ji}, Z_j ($j=1, \cdots, m$, $i=1, \cdots, k_j$) into a Banach space W, and f_{ji} are Lebesgue-Bochner summable functions, and μ_j are vector volumes.

The above integral is a generalization of the integral $\int u(f, d\mu)$ developed in [**1**]. An integral similar to the last integral, developed in a different way, one can find in Bourbaki [**10**, Chapter V, p. 48–49]. For applications, see the following paper in this volume.

1. **Properties of vector volumes.** Let R be the space of reals and Y_i, Z_i, W be seminormed spaces. Denote by $L(Y_1, \cdots, Y_k; W)$ the space of all k-linear continuous operators u from the space $Y_1 \times \cdots \times Y_k$ into the space W. The norms of elements in the above spaces will be denoted by $|\ |$.

The family of sets V of an abstract space X will be called a prering if for any two sets $A_1, A_2 \in V$ we have $A_1 \cap A_2 \in V$ and there exists disjoint sets $B_1, \cdots, B_k \in V$ such that $A_1 \backslash A_2 = B_1 \cup \cdots \cup B_k$.

A nonnegative function v on a prering V is called a positive volume or when there is no confusion just volume if it is countably additive, that is for every countable family of disjoint sets $A_t \in V$ $(t \in T)$ such that $A = \bigcup_T A_t \in V$ we have $v(A) = \sum_T v(A_t)$.

A function μ from a prering V into a Banach space Z is called a vector volume or simply volume when there is no confusion possible if the function μ is finite additive on V and for some positive volume v we have

$$|\mu(A)| \leqq v(A) \quad \text{for all } A \in V.$$

It follows from this definition and from the definition of a prering that every volume is countably additive.

THEOREM 1. *Let V_i be a prering of sets of a space X_i ($i=1, \cdots, k$). Denote by $V = V_1 \times \cdots \times V_k$ the family of all sets of the form $A = A_1 \times \cdots \times A_k$ where $A_i \in V_i$. Then V is a prering of sets of the space $X = X_1 \times \cdots \times X_k$.*

[1] This work was partially supported by NSF grant GP-2565.

A triple (X, V, v), where V is a prering of sets of the space X and v is a positive volume on the prering V will be called a volume space.

THEOREM 2. *Let (X_i, V_i, v_i) $(i=1, \cdots, k)$ be volume spaces. Then the triple (X, V, v), where $X=X_1\times \cdots \times X_k$, $V=V_1\times \cdots \times V_k$, and $v(A)=v_1(A_1) \cdots v_k(A_k)$ for $A=A_1\times \cdots \times A_k\in V$, is a volume space. The triple (X, V, v) will be called the product of the volume spaces (X_i, V_i, v_i).*

THEOREM 3. *Let V_i be a prering of sets of a space X_i $(i=1, \cdots, k)$. Let v be a positive volume on $V=V_1\times \cdots \times V_k$ and let $\bar{\mu}(A_1, A_2, \cdots A_k)$ be a function from the prering V into a Banach space Z finite additive with respect to every variable A_i separately. Then if*

$$|\bar{\mu}(A_1, \cdots, A_k)| \leqq v(A_1\times \cdots \times A_k) \quad \text{for all } A_1\times \cdots \times A_k \in V,$$

the function μ defined by the formula $\mu(A_1\times \cdots \times A_k)=\bar{\mu}(A_1, \cdots, A_k)$ is a vector volume on the prering V.

Let (X, V, v) be a fixed volume space. Denote by $M(v, Z)$ the set of all volumes μ from the prering V into the Banach space Z such that

$$|\mu(A)| \leqq cv(A) \quad \text{for all } A \in V.$$

The smallest constant satisfying the last inequality will be denoted by $\|\mu\|$. It is easy to see that the space $(M(v, Z), \|\ \|)$ is a Banach space.

THEOREM 4. *Let (X, V, v) be the product volume space of the volume spaces (X_i, V_i, v_i), $(i=1, \cdots, k)$. If $\mu_i\in M(v_i, Z_i)$ for $i=1, \cdots, k$ and $u\in L(Z_1, \cdots, Z_k; W)$ then $\mu\in M(v, W)$ and $\|\mu\|\leqq|u|\,\|\mu_1\| \cdots \|\mu_k\|$ where $\mu(A_1\times \cdots \times A_k)=u(\mu_1(A_1), \cdots, \mu_k(A_k))$ for all $A\in V$.*

The proof of the theorem follows immediately from the previous one.

2. Multilinear integrals and some relations between them.

LEMMA 1. *Let $(Y_i, |\ |_i)$ be a family of seminormed spaces and let E_i be a dense subspace of the space Y_i $(i=1, \cdots, k)$. If u is a k-linear operator from $E_1\times \cdots \times E_k$ into a Banach space W and*

$$|u(y_1, \cdots, y_k)| \leqq |u|\,|y_1|_1 \cdots |y_k|_k$$

for $y_i\in E_i$ $(i=1, \cdots, k)$ then the operator u has a unique extension to a k-linear operator u' such that $|u'(y_1, \cdots, y_k)|\leqq|u|\ |y_1|_1 \cdots |y_k|_k$ for $y_i\in Y_i$ $(i=1, \cdots, k)$.

Denote by $S(Y)$ the family of all functions of the form

$h=y_1\chi_{A_1}+\cdots+y_k\chi_{A_k}$, where $A_i\in V$ is a finite family of disjoint sets and $y_i\in Y$.

In [1] was developed the theory of the space $L(v, Y)$ of Lebesgue-Bochner summable functions f generated by a volume space (X, V, v) with values in a Banach space Y. The set $S(Y)$ according to Lemma 1 and Lemma 4, [1] is linear and dense in the space $L(v, Y)$.

Let

$$(X_{ji}, V_{ji}, v_{ji}) \quad (j = 1, \cdots, m; i = 1, \cdots, k_j)$$

be a family of volume spaces and let (X_j, V_j, v_j) be the product of the above volume spaces corresponding to a fixed j.

Let u be a multilinear continuous operator from the product of the Banach spaces Y_{ji}, Z_j $(j=1, \cdots, m; i=1, \cdots, k_j)$ into a Banach space W.

Let $\mu_j\in M(v_j, Z_j)$ and $s_{ji}\in S(Y_{ji})$. Take a representation

$$s_{ji} = \sum_{n_{ji}} y_{n_{ji}}\chi_{A_{n_{ji}}},$$

where

$$y_{n_{ji}} \in Y_{ji} \quad \text{and} \quad A_{n_{ji}} \in V_{ji}$$

are disjoint sets. Define

$$\int u(s_{ji}, d\mu_j) = \sum_j \sum_i \sum_{n_{ji}} u(y_{n_{ji}}, \mu_j(A_{n_{j1}} \times \cdots \times A_{n_{jk_j}})).$$

It is easy to see that the above operator is well defined, from the product of the spaces U, $S(Y_{ji})$, $M(v_j, Z_j)$ $(j=1, \cdots, m; i=1, \cdots, k_j)$ into the space W and

$$\left|\int u(s_{ji}, d\mu_j)\right| \leqq |u| \left(\prod_{ji} \|s_{ji}\|\right) \prod_j \|\mu_j\|$$

for all $u\in U$, $s_{ji}\in S(Y_{ji})$, $\mu_j\in M(v_j, Z_j)$.

Using Lemma 1 we can extend the above operator to an operator $\int u(f_{ji}, d\mu_j)$ defined on the product of the spaces U, $L(v_{ji}, y_{ji})$, $M(v_j, Z_j)$. Thus we have the following

THEOREM 5. *The operator* $\int u(f_{ji}, d\mu_j)$ *is multilinear from the product of the spaces* U, $L(v_{ji}, Y_{ji})$, $M(v_j, Z_j)$ $(j=1, \cdots, m; i=1, \cdots, k_j)$ *into the space* W *and*

$$\left|\int u(f_{ji}, d\mu_j)\right| \leqq |u| \left(\prod_{ji} \|f_{ji}\|\right)\left(\prod_j \|\mu_j\|\right)$$

for all $u \in U$, $f_{ji} \in L(v_{ji}, Y_{ji})$, $\mu_j \in M(v_j, Z_j)$.

THEOREM 6. *Let* (X, V, v) *be the product of volume spaces* (X_j, V_j, v_j) $(j=1, \cdots, k)$ *and let* $f_j \in L(v_j, Y_j)$. *Let* u *be a* k*-linear continuous operator from the space* $Y_1 \times \cdots \times Y_k$ *into* W. *Then the function* f *defined by the formula*

$$f(x_1, \cdots, x_k) = u(f_1(x_1), \cdots, f_k(x_k))$$

on the space X *belongs to the space* $L(v, W)$ *and*

$$\|f\| \leqq |u| \|f_1\| \cdots \|f_k\|.$$

Let (X, V, v) be the product of the volume spaces (X_j, V_j, v_j) where $j=1, \cdots, k$.

Let Y_j, Z be Banach spaces. Consider a multilinear operator u from the space $Y_1 \times \cdots \times Y_k \times Z$ into a Banach space W. Define a new operator u_0 from the space $Y_1 \times \cdots \times Y_k$ into the space $W_0 = L(Z; W)$ by means of the formula

$$u_0(y_1, \cdots, y_k)(z) = u(y_1, \cdots, y_k, z) \quad \text{for } y_i \in Y_i,\ z \in Z.$$

It is easy to see that the operator u_0 is k-linear and continuous.

Now if

$$f_j \in L(v_j, Y_j)$$

then according to the previous theorem we have

$$f = u_0(f_1, \cdots, f_k) \in L(v, W_0).$$

Define a new operator u_1 by means of the formula

$$u_1(w, z) = w(z) \quad \text{for } w \in W_0,\ z \in Z.$$

THEOREM 7. *If* $\mu \in M(v, Z)$ *and* $f = u_0(f_1, \cdots, f_k)$, u_0, u *are defined as above then*

$$\int u(f_1, \cdots, f_k, d\mu) = \int u_1(f, d\mu).$$

Now let Y_j, Z_j $(j=1, \cdots, k)$ be Banach spaces and let (X, V, v) be the product of the volume spaces (X_j, V_j, v_j) $(j=1, \cdots, k)$. Let

$$f_j \in L(v_j, Y_j) \quad \text{and} \quad \mu_j \in M(v_j, Z_j).$$

Consider a multilinear continuous operator u from the product of the spaces Y_j, Z_j $(j=1, \cdots, k)$ into a Banach space W. Let u_0 be an operator from the product of the spaces Z_j $(j=1, \cdots, k)$ into the space $W_0 = L(Y_1, \cdots, Y_k; W)$ defined by the formula

$$u_0(z_1, \cdots, z_k)(y_1, \cdots, y_k) = u(y_1, \cdots, y_k, z_1, \cdots z_k)$$

for $z_i \in Z_i$, $y_i \in Y_i$.

It is easy to see that the operator u_0 is k-linear and continuous. Thus from Theorem 4 we get

$$\mu = u_0(\mu_1, \cdots, \mu_k) \in M(v, W_0).$$

Let u_1 denote the multilinear continuous operator defined on the space $Y_1 \times \cdots \times Y_k \times W_0$ by means of the formula

$$u_1(y_1, \cdots, y_k, w) = w(y_1, \cdots, y_k) \quad \text{for } y_j \in Y_j, w \in W_0.$$

We have the following theorem.

THEOREM 8. *If* $f_j \in L(v_j, Y_j)$ *and* $\mu = u_0(\mu_1, \cdots, \mu_k)$, u_1 *are defined as above, then*

$$\int u(f_1, \cdots, f_k, d\mu_1, \cdots, d\mu_k) = \int u_1(f_1, \cdots, f_k, d\mu).$$

The last two theorems allow us to reduce any of the integrals to the following form $\int u(f, d\mu)$. In [5] has been shown how one can reduce the integrals to iterated integrals by means of generalized Fubini's Theorems.

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