Let $\lambda_1$ be the lowest eigenvalue of the membrane problem
\[
\Delta u + \lambda u = 0 \quad \text{in } D,
\]
\[u = 0 \quad \text{on } \partial D.
\]
It was shown by Barta [1] that if $w > 0$ in $D$, then
\[\lambda_1 \geq \inf \left[ -\frac{\Delta w}{w} \right].\]
This result has been extended to other self-adjoint problems for second order operators. See [2], [3], and [6].

The purpose of this note is to show that the same technique locates the spectrum of a nonself-adjoint problem in a half-plane. Such a result is of interest in investigating stability, where one needs to know whether there is any spectrum in the half-plane $\Re \lambda \leq 0$.

In a bounded domain $D$ we consider the differential equation
\[
L[u] + \lambda ku = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u + \lambda k(x)u
\]
(1)
\[= - k(x)f(x)\]
where $x \sim (x_1, \ldots, x_n)$. The matrix $a^{ij}(x)$ is symmetric and positive definite, $k(x)$ is positive, and all the coefficients are real and bounded in $D$. However, they need not be continuous.

The boundary $\partial D$ is divided into two disjoint parts $\Sigma_1$ and $\Sigma_2$, and the boundary conditions are
\[u = 0 \quad \text{on } \Sigma_1,
\]
(2)
\[M[u] = \sum_{i=1}^{n} e^i(x) \frac{\partial u}{\partial x_i} + g(x)u = 0 \quad \text{on } \Sigma_2.
\]
The vector field $e$ points outward from $D$.

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We shall prove the following theorem about the spectrum of the operator $L$ considered as an operator on the space $C(D)$ of continuous functions with the maximum norm.

**Theorem 1.** Suppose $w(x)$ defined on $D\cup \partial D$ has the properties:

(i) $w(x) > 0$ on $D\cup \partial D$;

(ii) $w \in C^2(D) \cap C^1(D\cup \partial D)$;

(iii) $M[w] \geq 0$ on $\Sigma_2$.

Then the discrete and continuous spectra of the problem (1), (2) are contained in the half-plane

$$
\Re \lambda \geq \inf \left( - \frac{L[w]}{kw} \right).
$$

**Proof.** Let $\tau = \inf( -L[w]/kw)$, and suppose that $\Re \lambda < \tau$. We wish to show that $\lambda$ is in the resolvent set.

Let $u$ satisfy (1) and (2), and define

$$
v(x) \equiv u(x)/w(x).
$$

Substituting $u = vw$ in (1), multiplying the equation by $\bar{v}$, and taking real parts, we obtain

$$
\sum_1^n \frac{1}{2} \omega a_{ij} \frac{\partial^2 |v|^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \frac{1}{2} \left( w b^i + 2 \sum_{j=1}^n a_{ij} \frac{\partial w}{\partial x_j} \right) \frac{\partial |v|^2}{\partial x_i} + (L[w] + \Re(\lambda)kw) |v|^2
\geq -k \Re(\bar{v})
$$

since $a_{ij}$ is positive definite. The boundary conditions yield

$$
|v|^2 = 0 \text{ on } \Sigma_1,
$$

$$
\sum_1^n \bar{v} \frac{\partial |v|^2}{\partial x_i} + 2M[w] |v|^2 = 0 \text{ on } \Sigma_2.
$$

We observe that $L[w] + \Re(\lambda)kw \leq -(\tau - \Re \lambda)kw$. Therefore by the maximum principle, we find that

$$
|v|^2 \leq \frac{1}{\tau - \Re \lambda} \sup_D \frac{\Re(\bar{v})}{w}.
$$

Hence
Thus if $\Re \lambda < \tau$, the operator $L + \lambda k$ has a bounded inverse in the maximum norm on its range. Hence $\lambda$ is in either the residual spectrum or the resolvent set. Therefore the discrete and continuous spectra are contained in the half-plane

$$\Re \lambda \geq \inf_D \left( - \frac{L[w]}{kw} \right)$$

as the theorem states.

In what follows we shall assume that the problem does not have a residual spectrum. That is, we assume that the range of $L + \lambda k$ is dense for some sufficiently small $\lambda$; or, equivalently, that the index is zero.

The following theorem shows that the bound (3) is a lower bound for a real point $\lambda_1$ of the spectrum:

**Theorem 2.** Suppose there is a function $w$ satisfying the conditions of Theorem 1. Then if the spectrum of (1), (2) is not empty, there exists a real number $\lambda_1$ in the spectrum such that the whole spectrum lies in the half-plane

$$\Re \lambda \geq \lambda_1.$$ 

**Proof.** Let $\lambda$ be real, and let $v = u/w$, where $u$ is real. Then the problem (1), (2) becomes

$$\sum_1^n w a_i v \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^n \left( w b_i + 2 \sum_{j=1}^n a_{ij} \frac{\partial w}{\partial x_j} \right) \frac{\partial v}{\partial x_i} + (L[w] + \lambda kw)v = - kf \text{ in } D,$n \quad v = 0 \text{ on } \Sigma_1,$

$$\sum_1^n w e_i \frac{\partial v}{\partial x_i} + M[w]v = 0 \text{ on } \Sigma_2.$$

By the maximum principle we see that if $\lambda < \inf(-L[w]/kw)$, then $f > 0$ implies $v > 0$ and hence $u > 0$. Thus the resolvent $R_\lambda$ is positive for $\lambda < \inf(-L[w]/kw)$. 

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Conversely if $R_r \geq 0$ for some real number $\tau$, we find that the solution $w$ of

$$L[w] + \tau kw = -k \quad \text{in } D,$$

$$w = 1 \quad \text{on } \Sigma_1,$$

$$M[w] = 0 \quad \text{on } \Sigma_2,$$

(4)

is admissible in Theorem 1, so that the spectrum lies in the half-plane $\Re \lambda \geq \tau$, and $R_\mu \geq 0$ for all real $\mu \leq \tau$.

Now let $\lambda_1$ be the limit superior of those $\lambda$ for which $R_\lambda \geq 0$. Then the spectrum is in the half-plane $\Re \lambda \geq \lambda_1$. If $\lambda_1$ is in the resolvent set, we see by continuity that $R_{\lambda_1} \geq 0$. Moreover, for any $\lambda > \lambda_1$ with $\lambda - \lambda_1 < \|R_{\lambda_1}\|^{-1}$ we have $R_\lambda = (I - (\lambda - \lambda_1)R_{\lambda_1})^{-1}R_{\lambda_1} = R_{\lambda_1} + (\lambda - \lambda_1)R_{\lambda_1}^2 + \cdots \geq 0$. Thus if $\lambda_1$ is in the resolvent set, we obtain a contradiction with the definition of $\lambda_1$. Hence $\lambda_1$ is in the spectrum of (1), (2).

We observe that for any $\tau < \lambda_1$ the solution $w$ of (4) gives the lower bound $\tau$, so that the lower bound (3) can be made arbitrarily close to $\lambda_1$ by a judicious choice of $w$.

**Remarks**

1. If $D$ is unbounded but $\Sigma_2$ is bounded, we can define a solution of (1), (2) by exhaustion. That is, we obtain the solutions $u_n$ of

$$L[u_n] + \lambda ku_n = -k \quad \text{in } D \cap \{|x| < n\},$$

$$u_n = 0 \quad \text{on } \Sigma_1 \cup \{|x| = n\},$$

$$M[u_n] = 0 \quad \text{on } \Sigma_2.$$

By the method used in the proof of Theorem 1 we find that if $\Re \lambda < \inf(-L[w]/kw)$, the functions $u_n$ converge uniformly to a solution $u$ of (1), (2). Thus the spectrum still lies in $\Re \lambda \geq \inf(-L[w]/kw)$. Theorem 2 can also be extended to this case.

2. If $D$ and the coefficients of our problem are so smooth that for sufficiently small real $\mu$ the resolvent $R_\mu$ is completely continuous in the maximum norm (i.e., the family $R_\mu[f]$ with $f \leq 1$ is equicontinuous), then the spectrum is discrete, so that $\lambda_1$ is an eigenvalue.

A theorem of Krein and Rutman [5, Theorem 6.1] shows that the corresponding eigenfunction $u_1$ is positive in $D$. The theorem of Krein and Rutman also states that in this case the adjoint operator $R^*_\mu$ has the eigenvalue $(\lambda_1 - \mu)^{-1}$ with a positive eigenfunctional $u^*_1$. From this fact we can derive Theorem 1 with condition (i) replaced by the weaker condition $w \geq 0$. Moreover, we can obtain a complementary upper bound for $\lambda_1$.
If \( q(x) \geq 0 \) in \( D \), \( q = 0 \) on \( \Sigma_1 \), and \( M[q] \leq 0 \) on \( \Sigma_2 \), then \( \lambda_1 \leq \sup (-L[q]/kq) \).

3. If the coefficients are so smooth that the adjoint operator \( L^* \) can be formed, and if the boundary conditions are self-adjoint (e.g., \( \Sigma_1 = \partial D \)), an inequality of the same type as (3) may be found by methods of Hooker [4] and Protter [6]. Namely,

\[
\text{Re}(\lambda) \geq \inf_D \left( - \frac{L[w] + L^*[w]}{2kw} \right).
\]

This inequality may be stronger or weaker than (3).

**BIBLIOGRAPHY**


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