

**SOME RESULTS GIVING RATES OF CONVERGENCE IN
THE LAW OF LARGE NUMBERS FOR WEIGHTED
SUMS OF INDEPENDENT RANDOM VARIABLES**

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Communicated by J. L. Doob, October 8, 1965

Let $\{X_N\}$ for $N=1, 2, \dots$ be an independent sequence of random variables, and let $S_N = X_1 + \dots + X_N$. Probabilists have expended considerable effort investigating the convergence of $\{(S_N - b_N)/a_N\}$ where $\{b_N\}$ and $\{a_N\}$ are sequences of centering and weighting constants respectively. Recently Baum and Katz in [1], [2], and [3] have investigated the rate of convergence to zero of appropriately normalized sums, obtaining (along with some other results) results on the convergence of series of the form $\sum N^\gamma P\{|S_N - N\mu| > N^\beta\}$ where the X 's are assumed to be identically distributed with common mean μ . Pruitt in [4] obtained a sufficient condition for sums of the form $S_N = \sum_k a_{N,k} X_k$ to converge to μ .

Now let $\{X_k\}$ be an independent sequence of random variables having finite first moments and define

$$F(y) = \sup_k P\{|X_k - EX_k| > y\}.$$

Let $\{a_{N,k}\}$ for $N, k=1, 2, \dots$ be real numbers such that

- (1) $\max_k |a_{N,k}| = CN^{-\beta},$
- (2) $\sum_k |a_{N,k}| \leq CN^\alpha,$
- (3) $\sum_k |a_{N,k}|^t \leq CN^{-\rho}.$

Define

$$S_N = \sum_k a_{N,k}(X_k - EX_k).$$

We have obtained the following five theorems.

THEOREM 1. *If $\rho > 0, \beta > 0, \alpha < \beta, t > 1$, and $y^t F(y) \leq M < \infty$ for all $y > 0$, then for every $\epsilon > 0$*

$$P\{|S_N| > \epsilon\} \leq O(N^{-\rho}).$$

¹ Research supported by the Air Force Office of Scientific Research.

THEOREM 2. *If $\rho > 0, \beta > 0, \alpha < \beta, t > 1$, and $y^t F(y) \rightarrow 0$ as $y \rightarrow \infty$, then for every $\epsilon > 0$*

$$P\{|S_N| > \epsilon\} = o(N^{-\rho}).$$

THEOREM 3. *If $\beta(t-1) - \alpha > 0, \beta > 0, \alpha < \beta, t > 1$, and F satisfies*

$$\lim_{y \rightarrow \infty} F(y) = 0 \quad \text{and} \quad \int y^t |dF(y)| < \infty,$$

then for every $\epsilon > 0$

$$\sum_N N^{\beta(t-1) - \alpha - 1} P\{|S_N| > \epsilon\} < \infty.$$

THEOREM 4. *If $\rho > 0, \beta > 0, \alpha < \beta, t \geq 1$, and there exists a nonnegative and nonincreasing real-valued function $G(x) \geq F(x)$ satisfying*

$$\lim_{y \rightarrow \infty} G(y) = 0 \quad \text{and} \quad \int_0^\infty y^t |dG(y)| < \infty$$

such that

$$(4) \quad \sup_{x \geq 1} \sup_{y \geq x} \frac{y^t F(y)}{x^t G(x)} = \gamma < \infty,$$

then for every $\epsilon > 0$

$$(5) \quad \sum_N N^{\rho-1} P\{|S_N| > \epsilon\} < \infty.$$

THEOREM 5. *If $\rho > 0, \beta > 0, \alpha < \beta, t \geq 1$, and F satisfies*

$$\lim_{y \rightarrow \infty} F(y) = 0 \quad \text{and} \quad \int_0^\infty y^t \log^+ y |dF(y)| < \infty,$$

then (5) holds for every $\epsilon > 0$.

One should immediately notice that for $t \geq 1$ we have

$$\begin{aligned} \sum_k |a_{N,k}|^t &\leq \left(\max_k |a_{N,k}|\right)^{t-1} \sum_k |a_{N,k}| \\ &\leq C^2 N^{\alpha - \beta(t-1)} \end{aligned}$$

so that ρ can be assumed to be at least as large as $\beta(t-1) - \alpha$. Note that Theorem 1 implies $\sum_N N^{\rho-1-\delta} P\{|S_N| > \epsilon\} < \infty$ for every $\delta > 0$ so that the additional assumptions used in Theorems 4 and 5 do not give very much more than that already obtained in Theorem 1. The

assumption (4) can readily be violated but most "reasonable" distributions will satisfy it.

Though Theorems 4 and 5 are considerably stronger than Theorem 3, two known results can be obtained as corollaries of Theorem 3 by specializing the constant t and the constants α and β from (1) and (2). Theorem 2 of [4] is obtained by setting $t = 1 + 1/\gamma$, $\alpha = 0$, and $\beta = \gamma$; a part of Theorem 3 of [2] and [3] is obtained by leaving t as is and setting $\alpha = 1 - r/t$ and $\beta = r/t$ with $\frac{1}{2} < r/t \leq 1$.

The motivation for this work was as follows. The average $\frac{1}{2}X_1 + \frac{1}{4}X_2 + \frac{1}{4}X_3$ is at least as "fine" an average as is $\frac{1}{2}X_1 + \frac{1}{2}X_2$ and in some sense the first average is "finer" than the second. It has always seemed reasonable to the authors that a finer average than the standard average $(1/N)\{X_1 + \cdots + X_N\}$ should not hurt convergence any and might actually improve the rate of convergence if one could use the right quantitative measure of the improvement in averaging. The exponent ρ used in (3) seems to be the correct measure of averaging to use.

The methods used in the proof of these theorems apparently originated with Erdős [5]. The method was modified and improved by Katz [1] and modified still more by Pruitt [4]. Detailed proofs will appear elsewhere.

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