1. Introduction. J. Tate [8] has introduced a theory of cohomological dimension for fields using the étale Grothendieck (=Galois) cohomology. In recent work, M. Artin has extended these methods to produce a dimension theory for noetherian preschemes. On the other hand, the author [5] has used the flat Grothendieck cohomology over a field to study certain duality questions (see also [7], [9] for the étale case); so it is natural to ask whether there exists a dimension theory based on the flat cohomology. We shall show that the answer is, in general, no. Full proofs will appear in [6].

2. Terminology. A Grothendieck topology is a pair consisting of a category $\text{Cat}_T$ and a set $\text{Cov}_T$ of families of morphisms of $\text{Cat}_T$. They are subjected to the axioms:

1. If $\phi$ is an isomorphism, $\{\phi\} \in \text{Cov}_T$.
2. If $\{U_i \to U\} \in \text{Cov}_T$ and $\{V_{ij} \to U_i\} \in \text{Cov}_T$, for all $i$, then $\{V_{ij} \to U\} \in \text{Cov}_T$.
3. If $\{U_i \to U\} \in \text{Cov}_T$ and $V \to U$ is arbitrary, then $U_i \times_U V$ exists for each $i$, and $\{U_i \times_U V\} \in \text{Cov}_T$.

A presheaf (of abelian groups) on $T$ is a contravariant functor from $\text{Cat}_T$ to the category of abelian groups, while a sheaf, $F$, is a presheaf which satisfies the axiom

\[ F(U) \to \prod_i F(U_i) \to \bigoplus_{i,j} F(U_i \times_U U_j) \]

is exact (i.e., $F(U)$ is mapped bijectively onto the set of all $x \in \prod_i F(U_i)$ whose images by the two maps shown agree in $\bigoplus_{i,j} F(U_i \times_U U_j)$.) Roughly speaking, all that is done in Godement's book [2] for classical sheaf theory may be done in this general setting [1]. If $X$ is a prescheme [3, Vol. I, p. 97], we let $\text{Cat}_T$ be the category of all preschemes $Y$ which are separated, finitely presented, flat, and quasi-finite over $X$ [3, Vol. I, p. 135, p. 144; Vol. IV, p. 5; Vol. II, p. 115]. $\text{Cov}_T$ consists of arbitrary families of flat morphisms whose disjoint sum is faithfully flat [3, Vol. IV, Part 2, p. 9]. It is known that these

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data define a Grothendieck topology, which we call the flat topology on $X$.

3. Main results.

**Theorem 1.** Let $X = \text{Spec } A$, with $A$ a complete, noetherian local ring of characteristic $p > 0$. Unless $A$ is a perfect field, we have $c.d._p X = \infty$; that is, for every integer $n \geq 0$, there exists a torsion sheaf $F_n$ in the flat topology on $X$ such that the $p$-primary component of $H^n(X, F_n)$ is not zero. Moreover, if the residue field of $A$ is separably closed, then the sheaves $F_n$ (for $n > 0$) may be chosen so that

\[
H^r(X, F_n) = (0) \text{ for } r \neq 0, \ r \neq n
\]

(*)

\[
H^n(X, F_n) = A^+/A^{+p}.
\]

**Corollary.** Let $k$ be a field of characteristic $p > 0$. Then the following statements are equivalent:

1. $k$ is perfect,
2. $c.d._p k \leq 1$,
3. $c.d._p k$ is finite,
4. $c.d._p k_* = 0$,
5. $c.d._p k_*$ is finite.

**Theorem 2.** Let $k$ be a field of characteristic $p > 0$. Let $G$ be a commutative group scheme \([3, \text{Vol. II, p. 166}], [5, \text{p. 412}]\) of finite type over $k$. Then for every $r > 2$ we have $H^r(k, G; p) = (0)$. (Here, $H^r(X, F; p)$ denotes the $p$-primary component of the group $H^r(X, F)$.) Consequently, by restricting the coefficient category to those sheaves which are representable (or their limits) we may bound the $p$-dimension of the field $k$ by 2.

4. Sketch of proofs. Over a complete local ring, one may replace quasi-finite by finite with no change in the dimension theory. Every finite algebra over $A$ is a complete semilocal ring, hence a direct product of complete local rings. Thus every object of $\text{Cat } T$ is uniquely a sum of connected schemes and all constructions and verifications may be restricted to the connected objects of $\text{Cat } T$. In the case of separably closed residue field, for each abelian group $G$ and each sheaf of sets $F$ over $X$, we define a presheaf $\alpha_F$ by

\[
\alpha_F(U) = \coprod_{F(U)} \ G, \ U \text{ connected}
\]

where $\coprod_{F(U)} \ G$ means the direct sum of copies of $G$ indexed by the set $F(U)$, and we extend the definition of $\alpha_F$ in the usual way to the nonconnected objects of $\text{Cat } T$.

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*That is, a sheaf whose values lie in torsion abelian groups.*
Lemma. Let $A$ be as in Theorem 1 with separably closed residue field. Then the presheaves $\mathcal{A}_F$ are sheaves and for every object $U$ and covering $V$ of $U$ in $\text{Cat } T$,

$$H^r(V/U, \mathcal{A}_F) = (0) \text{ for } r > 0.$$  

From equation (**), one deduces that $H^r(U, \mathcal{A}_F) = (0)$ for every $r > 0$ and every object $U$ of $\text{Cat } T$. If $F_1 = a_p$, where $a_p$ is the kernel of the Frobenius map on the additive group scheme $G_a$, then $F_1$ satisfies equation (*). The exact sequence

$$0 \rightarrow F_2 \rightarrow (\mathbb{Z}/p\mathbb{Z})_{F_1} \rightarrow F_1 \rightarrow 0$$

and the above lemma, show that $F_2$ satisfies equation (*). One now proceeds by induction using the lemma and the exact sequence

$$0 \rightarrow F_{n+1} \rightarrow (\mathbb{Z}/p\mathbb{Z})_{F_n} \rightarrow F_n \rightarrow 0.$$  

For the general case, one analyzes the Leray spectral sequence [1]

$$H^s(X, R^v\pi_*F) \Rightarrow H^*(X, F)$$

(where $X = \text{Spec } A \otimes_k k_s$, $k = \text{residue field of } A$, $k_s = \text{separable closure of } k$).

The Corollary follows immediately from the theorem if one uses the Hochschild-Serre spectral sequence [1, p. 92] as applied to Grothendieck cohomology.

Theorem 2 is proved by reducing it to a question concerning artinian group schemes [5, pp. 412–413]. This is done via a structure theorem for the category of sheaves over $k$, and results of Tate. The conclusion is: in order to prove Theorem 2 it suffices to prove it for the kernel $G_n$ of the $n$-th iterate of the Frobenius map on $G$. In this case, we make use of the Hochschild-Serre spectral sequence and the structure of a composition series for $G_n$ over $k_s$ to reduce the theorem to two assertions

(i) $H^r(k_s, a_p) = (0)$ for $r > 1$,

(ii) $H^r(k_s, u_p) = (0)$ for $r > 1$.

Here $u_p$ is the kernel of the Frobenius map on the multiplicative group scheme $G_m$. Assertion (i) is known from [4, p. 21], [5, Proposition 3], and assertion (ii) follows because one can prove

$$H^r(k_s, G_m) = (0) \text{ for } r > 0.$$  

References


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