ON A FIXED POINT THEOREM FOR NONLINEAR P-COMPACT OPERATORS IN BANACH SPACE[1]

BY W. V. PETRYSHYN

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1. **Introduction.** In [5] Kaniel proved a fixed point theorem for a nonlinear quasi-compact operator in a Banach space. The purpose of this paper is to generalize and simplify Kaniel's main result and its proof to a more general class of nonlinear operators which we call projectionally-compact (P-compact) and which, among others, contains completely continuous, quasi-compact, and monotone operators. From our general fixed point theorem for P-compact operators we then deduce in a simple way the fixed point theorems of Schauder [11], Rothe [10], Krasnoselsky [6], Altman [1], Kaniel [5], and others. In case the underlying space is a Hilbert space, we deduce (see also Kaniel [5]) some theorems on strongly monotone operators obtained by Minty [7] and Browder [2], [3], [4]. Let us add that our conditions have a form which not only admits a simpler investigation but at the same time seem to be more natural and suitable for applications to numerical functional analysis. Furthermore, the method of our proof is basically constructive. In fact, we show in [9] that it is essentially the projection method of which the Galerkin method is one of its simplest realizations. The latter methods are known [8] to play an important role in the approximate solution of operator equations.

2. **Preliminary results.** Let X be a finite dimensional Banach space; let B_r denote the closed ball in X of radius $r>0$ about the origin and let S_r denote the boundary of B_r. For later use we state first an essentially known fixed point theorem whose brief proof, which is based on the Brouwer fixed point theorem and the retraction mapping principle is given in [9].

THEOREM 1. *Let A be a continuous mapping of B_r into X and let μ be any constant. Then there exists at least one element u in $B_r - S_r$ such that*

$$Au - \mu u = 0 \tag{1}$$

provided that the mapping A satisfies the condition:

(π_μ): *If for some x in S_r the equation $Ax = \alpha x$ holds then $\alpha < \mu$.*

[1] The expanded version of this paper with detailed proofs will appear in [9].

Let us note that our Theorem 1 is equivalent to Theorem 1 proved in [5] by a more complicated argument. For other special cases of Theorem 1 see [9].

3. **Main results for infinite dimensional spaces.** In order to generalize Theorem 1 to operators defined on infinite dimensional real Banach and Hilbert spaces we assume that the Banach space X has the property that there exists a sequence $\{X_n\}$ of finite dimensional subspaces X_n of X, a sequence of linear projections $\{P_n\}$ on X, and a constant $K>0$ such that

$$(2) \qquad P_nX = X_n, \quad X_n \subset X_{n+1}, \quad n = 1, 2, 3, \cdots \quad \overline{\bigcup_n^\infty X_n} = X,$$

$$(3) \qquad \|P_n\| \leqq K, \quad n = 1, 2, 3, \cdots.$$

REMARK 1. In this paper we shall use the symbols "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in X, respectively.

For operators A defined on X or on a subset of X, we consider here only those operators A which are *bounded*,[2] i.e., operators which map bounded sets in X into bounded sets in X.

DEFINITION 1. *A nonlinear operator A will be called P-compact if P_nA is continuous in X_n for all sufficiently large n and if for any constant $p>0$ and any bounded sequence $\{x_n\}$ with $x_n \in X_n$ the strong convergence of the sequence $\{g_n\} \equiv \{P_nAx_n - px_n\}$ implies the existence of a strongly convergent subsequence $\{x_{n_i}\}$ and an element x in X such that $x_{n_i} \to x$ and $P_{n_i}Ax_{n_i} \to Ax$, as $n_i \to \infty$.*

For this class of operators the following fixed point theorem is valid.

THEOREM 2. *Suppose that A is P-compact. Suppose further that for given $r>0$ and $\mu>0$ the operator A satisfies the condition*:

(π_μ): *If some some x in S_r the equation $Ax = \alpha x$ holds then $\alpha < \mu$; then there exists at least one element u in $(B_r - S_r)$ such that*

$$(4) \qquad Au - \mu u = 0.$$

PROOF. The proof of Theorem 2 depends on Theorem 1 and the following lemma.

LEMMA 1. *If A satisfies the conditions of Theorem 2, then there exists an integer $n_0>0$ such that if $n \geqq n_0$ and $P_nAx = \beta x$ for some x in $S_r \cap X_n$ then $\beta < \mu$.*

PROOF OF LEMMA 1. Let us first note that in view of (3) and the

[2] The much more general results analogous to Theorems 2 and 4 below (to be published elsewhere) were since obtained by the author without the assumption that A be bounded.

boundedness of A there exists a constant $c>0$ such that $\|P_nAx\| \leqq c$ for all x in B_r. If the assertion of Lemma 1 were not true, we could find a sequence $\{x_n\}$ with $x_n \in X_n \cap S_r$ and a sequence of numbers $\{\beta_n\}$ such that

$$P_nAx_n = \beta_n x_n, \qquad (\beta_n \geqq \mu). \tag{5}$$

Hence

$$\beta_n r = \|\beta_n x_n\| = \|P_nAx_n\| \leqq c,$$

i.e., $\beta_n \in [\mu, c/r]$ for each n. Passing to a subsequence, we may assume that $\beta_n \to \beta$ and $\beta \in [\mu, c/r]$. This and (5) imply that

$$P_nAx_n - \beta x_n = (\beta_n - \beta)x_n \to 0, \qquad (n \to \infty). \tag{6}$$

Since A is P-compact, (6) implies the existence of a strongly convergent subsequence, which we again denote by $\{x_n\}$, and an element x in $S_r \cap X$ such that

$$x_n \to x \quad \text{and} \quad P_nAx_n \to Ax. \tag{7}$$

This and (6) imply that $Ax - \beta x = 0$ for $x \in S_r$ and $\beta \geqq \mu$ in contradiction to the condition (π_μ) of Theorem 2.

Proof of Theorem 2 completed. By Lemma 1, we can use Theorem 1 for the finite dimensional spaces X_n and the operators P_nA. Consequently, there exists an integer $N_0 > 0$ such that for each $n \geqq N_0$ there exists at least one element u_n in $B_r \cap X_n$ such that

$$P_nAu_n - \mu u_n = 0. \tag{8}$$

Therefore, again by the P-compactness of A, there exists a subsequence again denoted by $\{u_n\}$ and an element $u \in B_r \cap X$ such that $u_n \to u$, $P_nAu_n \to Au$, and $Au - \mu u = 0$. The last equation implies that $u \in (B_r - S_r)$ since the assumption that $u \in S_r$ would lead to the contradiction of condition (π_μ).

Remark 2. It is obvious that if in Definition 1 we require $p<0$ instead of $p>0$, then we get a theorem analogous to Theorem 2. We need only consider $-A$ instead of A and assume that for some $r>0$ and any $\mu<0$ instead of the condition (π_μ) the operator A satisfies the condition (π_μ^-): If for some x in S_r the equation $Ax = \alpha x$ holds then $\alpha > \mu$.

To see what type of operators belong to the class of P-compact operators let us first recall (following [3], [4], [5], [7]) that:

A is *demicontinuous* if $x_n \to x$ implies $Ax_n \rightharpoonup Ax$;

A is *quasi-compact* if A satisfies the following conditions: (i) A is bounded, (ii) $x_n \to x$ implies $P_mAx_n \to P_mAx$ for $m = 1, 2, 3, 4, \cdots$, (iii) if for some $\lambda > 0$ the sequence $\{g_n\} \equiv \{Ax_n + \lambda x_n\}$, where x_n is a bounded sequence, is strongly convergent, then there exists a strongly convergent subsequence $\{x_{n_i}\}$ of $\{x_n\}$, (iv) if for some $\lambda > 0$ the sequence $\{g_n\} \equiv \{P_n A x_n + \lambda x_n\}$, where x_n is a bounded sequence with $x_n \in X_n$ is strongly convergent, then there exists a subsequence $\{x_{n_i}\}$ which is strongly convergent;

A is *strongly continuous* if $x_n \rightharpoonup x$ implies $Ax_n \to Ax$;

A is *monotone increasing* on the Hilbert space $X = H$ if $(Ax - Ay, x - y) \geqq 0$ for all x and y in H;

A is *monotone decreasing* if $-A$ is monotone increasing;

A is *hemicontinuous* if it is continuous from each line segment in H to the weak topology in H. The following theorem whose simple proof is given in [9] specifies the relation between the operators defined above and the class of P-compact operators.

THEOREM 3. *The class of P-compact operators with $p < 0$ includes:*

(a) *Completely continuous and strongly continuous operators on X.*

(b) *Quasi-compact operators in X.*

(c) *Hemicontinuous (and hence demicontinuous, continuous and weakly continuous) monotone increasing operators in $X = H$.*

THEOREM 4. *Let A be P-compact and suppose that there exists a sequence of spheres S_{r_n} and a sequence of positive numbers $k_n \to \infty$ such that for any $\eta \geqq \mu > 0$ and any $x \in S_{r_n}$*

$$\|Ax - \eta x\| \geqq k_n. \tag{9}$$

Then for every $f \in X$ there exists an element x which satisfies the equation

$$Ax - \mu x = f. \tag{10}$$

4. **Special cases.** In this section we show that for the Banach space X satisfying (2) and (3) many of the known fixed point theorems are special cases of Theorem 2. Thus, we supply elementary and essentially constructive proofs of these theorems.

THEOREM S (SCHAUDER). *If A is a completely continuous mapping of B_r into B_r, then A has a fixed point in B_r.*

PROOF. We may assume, without loss of generality, that A has no fixed points on S_r. First, by Theorem 3(a), A is P-compact and second if $Ax = \alpha x$ for some x in S_r, then the condition $A(B_r) \subset B_r$ implies that $\alpha < 1$. Hence by Theorem 2, A has a fixed point in $(B_r - S_r)$.

In a similar way we also deduce[1] from Theorem 2 the following:

THEOREM R (ROTHE). *If A is a completely continuous mapping of B_r into X such that $A(S_r) \subset B_r$, then A has a fixed point in B_r.*

THEOREM A (ALTMAN). *If A is a completely continuous mapping of B_r into X such that $\|Ax-x\|^2 \geqq \|Ax\|^2 - \|x\|^2$ for all x in S_r, then A has a fixed point in B_r.*

If X is a Hilbert space, Theorem A was first proved by Krasnoselsky [6].

THEOREM K (KANIEL). *If A is a quasi-compact mapping of B_r into X such that $Ax+\lambda x \neq 0$ for all x in S_r and any $\lambda > \mu > 0$, then there exists an element u in B_r such that $Au+\mu u=0$.*

The following apparently new comparison theorem is also valid.

THEOREM 5. *If A and B are two P-compact mappings of B_r into H such that*

$$(11) \quad (Ax, x) \leqq \|x\|^2 \quad \text{and} \quad \|Ax - Bx\| \leqq \|x - Ax\| \quad \text{for all } x \text{ in } S_r,$$

then B has a fixed point in B_r.

If X is a Hilbert space H and A is a monotone decreasing operator on H, then one of the first important theorems of Minty [7] says essentially that if A is continuous and $\mu > 0$ then the operator $(A - \mu I)$ is onto. In [3], [4] Browder has proved this theorem (as well as other more general theorems) for demicontinuous and hemicontinuous operators while Shinbrot [12] established its validity for weakly continuous operators A. In view of our Theorem 3, Remark 2, and the fact noted in [5] that monotone increasing operators $C = -A$ satisfy the condition (9), it follows[1] that these particular results of all of the above authors are immediately deducible from our Theorem 4.

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UNIVERSITY OF CHICAGO

A PERTURBATION LEMMA

BY KARL GUSTAFSON[1]

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1. **Introduction.** We will prove the following lemma and investigate some of its implications: namely, a short proof by Goldberg [1] of the basic perturbation theorem of Kato [2], avoiding previous homotopy arguments; an extension of results of Trotter and Nelson [3] for semigroup generators; and a criterion for well-posed perturbed problems in spaces that are not necessarily complete. For further references and more information, see [1], [2], and [3].

Throughout this paper all operators are linear with domains subspaces of a normed linear space X and ranges subspaces of a normed linear space Y. If an operator B perturbs an operator T, we assume that $D(B)\supset D(T)$.

In this section, the spaces need not be complete.

LEMMA 1. *Let T^{-1} and B be bounded operators with $\|B\|<\|T^{-1}\|^{-1}$. Then*

$$\dim Y/\mathrm{Cl}\,(R(T)) = \dim Y/\mathrm{Cl}\,(R(T+B)). \tag{1.1}$$

PROOF.[2] We use the known result (e.g., see [1] for a proof) that if $\|B\|<\|T^{-1}\|^{-1}$, then

$$\dim Y/\mathrm{Cl}\,(R(T+B)) \leqq \dim Y/\mathrm{Cl}\,(R(T)). \tag{1.2}$$

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[2] Concerning this little result, let $\|B\|<\alpha\|T^{-1}\|^{-1}$. The author appreciates discussions with Dr. Seymour Goldberg, who proved it for $\alpha=1/2$ in his lectures. The main trick in the proof can be seen for the case $\alpha=3/4$. The author also appreciates the aid of Mr J. Kuttler in extending the result from $\alpha=3/4$ to $\alpha=7/8$.