IDEALS WITH SMALL AUTOMORPHISMS

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In [1], Forelli proves the following: If $G_1$ and $G_2$ are locally compact Abelian groups, if $J$ is a closed ideal in the group algebra $L^1(G_1)$, and if $\Psi$ is a homomorphism of $J$ into the measure algebra $M(G_2)$ with $\|\Psi\| = 1$, then $\Psi$ is induced by an affine map of a coset in $\Gamma_2$ into $\Gamma_1$. (See [1] for a more detailed statement. For notation and terminology, see [1] or [2]; $\Gamma_i$ denotes the dual group of $G_i$; the circle group will be denoted by $T$.) As Forelli points out in [1], the assumption $\|\Psi\| = 1$ cannot be entirely discarded.

Actually, the assumption $\|\Psi\| = 1$ cannot even be replaced by $\|\Psi\| < 1 + \epsilon$, no matter how small $\epsilon > 0$ is, even if “affine” is replaced by “piecewise affine” in the conclusion, and even if $G_1 = G_2 = T$ and $\Phi$ is assumed to be one-to-one.

Since the integer group $\mathbb{Z}$ admits only countably many piecewise affine maps, the preceding statement is a consequence of the theorem below. By way of contrast, it may be of interest to mention that if $\Psi$ is a homomorphism of all of $L^1(G_1)$ into $M(G_2)$ and if $\|\Psi\| > 1$, then $\|\Psi\| \geq \sqrt{5}/2$ [2, p. 88].

**Theorem.** Suppose $0 < \epsilon < 1$. Let $E$ be a set of positive integers $\lambda_k$ such that $\lambda_1 = 1$ and

$$
\sum_{k=1}^{\infty} \frac{\lambda_k}{\lambda_{k+1}} < \frac{\epsilon}{6\pi}.
$$

Let $J$ be the set of all $f \in L^1(T)$ whose $n$th Fourier coefficient $\hat{f}(n)$ is 0 for all $n$ not in $E$. Then $J$ is a closed ideal in $L^1(T)$, with continuum many automorphisms, and every automorphism $A$ of $J$ (other than the identity) satisfies the inequality

$$
1 < \|A\| < 1 + \epsilon.
$$

We shall sketch the proof.

Each $A$ is induced by a permutation $\alpha$ of $E$. The gaps in $E$ show that no affine map (other than the identity) carries $E$ onto $E$. Thus $\|A\| > 1$ if $A \neq I$.

We write $e(t)$ in place of $e^{2\pi it}$.

Let $\alpha$ be any permutation of $\mathbb{Z}^+$ (the positive integers), let

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be trigonometric polynomials in $J$. The theorem is an immediate consequence of the inequality

$$
\|g\|_1 \leq \left(1 + \frac{7\epsilon}{9}\right)\|f\|_1.
$$

To prove (4), fix $N$ so that $f$ and $g$ have their supports in $\{\lambda_1, \cdots, \lambda_N\}$. For $0 \leq t < 1$, let $D_t$ be the set of all $x = (x_1, \cdots, x_N)$ in $\mathbb{R}^N$ such that

$$
\lambda_{k-1} < x_k < \lambda_k + \frac{\lambda_{k+1}}{\lambda_k} \quad (1 \leq k \leq N - 1),
$$

and let $Q$ be the union of these $(N-1)$-cells $D_t$. We claim that $Q$ contains no point of $Z^N$, except 0: Assume, to get a contradiction, that $x \in Q \cap Z^N$, $x \neq 0$.

If $x_1 = 0$, (5) implies $t = 0$, hence $x_2 = \cdots = x_N = 0$. So $x_1 = 1 = \lambda_1$. If $2 \leq k \leq N$ and $x_{k-1} = \lambda_{k-1}$, (5) gives

$$
\lambda_{k-1} < \lambda_{k-1} + \frac{\lambda_{k+1}}{\lambda_k} \leq \lambda_{k-1}(1 + x_k)/\lambda_k,
$$
or $\lambda_k < 1 + x_k$. Since $x_k \leq \lambda_k$, we have $x_k = \lambda_k$. This leads to $x_N = \lambda_N$, a contradiction to the last equation (5).

Since $Q$ is a parallelepiped with one vertex at 0 it now follows that no two points of $Q$ are congruent modulo $Z^N$. Also, $Q$ has volume 1. Thus if we regard functions on the torus $T^N$ as periodic functions on $\mathbb{R}^N$, with period 1 in each of the variables $x_1, \cdots, x_N$, integration over $T^N$ may be replaced by integration over $Q$.

We return to our polynomials (3) and define

$$
F(x) = \sum_{k=1}^{N} c(k)e(\lambda_i t), \quad G(x) = \sum_{k=1}^{N} c(\alpha(k))e(\lambda_i k) \quad (x \in \mathbb{R}^N).
$$

These are trigonometric polynomials on $T^N$. Clearly

$$
\|F\|_1 = \|G\|_1.
$$

For $x \in D_t$, put $\tilde{F}(x) = f(t), \tilde{G}(x) = g(t)$. This defines $\tilde{F}$ on $Q$, hence on $T^N$; (5) and (1) imply that

$$
|\tilde{F}(x) - F(x)| \leq \sum_{k=1}^{N} |c(k)| \left| e(\lambda_{k-1} t) - e(x_k) \right|
$$

$$
\leq 2\pi \sum_{k=1}^{N} |c(k)| \lambda_k/\lambda_{k+1} \leq \frac{\epsilon}{3} \|f\|_1.
$$
if \( x \in D_t \). Since \( \|F\|_1 \) can be computed by integrating \( |F| \) over \( Q \), the definition of \( F \) shows, via Fubini's theorem, that \( \|F\|_1 = \|f\|_1 \). By (8) this gives

(9) \[ \|F\|_1 \leq \left( 1 + \frac{\epsilon}{3} \right) \|f\|_1. \]

The inequality \( \|g\|_1 \leq (1 + \epsilon/3) \|G\|_1 \) is obtained in the same way; combined with (7) and (9) it yields (4).

Remark. If \( E = \{\lambda_k\} \) is as in the theorem, if \( 1 \leq p \leq \infty \), if \( \sum c(k)e(\lambda_k t) \) is the Fourier series of some \( f \in L^p(T) \), and if \( \alpha \) is any permutation of \( Z^+ \), the above proof also shows that \( \sum c(\alpha(k))e(\lambda_k t) \) is the Fourier series of a function \( g \in L^p(T) \), and that \( \|g\|_p \leq (1 + \epsilon)\|f\|_p \).

References