BOOK REVIEW

Algebraic number theory. By Edwin Weiss. McGraw-Hill, New York, 1963. xii+275 pp. $9.95.

One can initiate the study of algebraic number theory either "globally" or "locally," i.e., either by considering ideals in the rings of integers of number fields, or else by looking first at the behavior of field extensions at a single prime divisor and then investigating the relationships among different primes of the same field. Such a choice of approaches is not too surprising, since the subject occupies a position midway between pure algebra and orthodox number theory, and in fact has given rise to two distinct treatments of the foundations of algebraic number theory, called by Hasse "ideal-theoretic" and "divisor-theoretic," respectively.

The ideal-theoretic approach has enjoyed much greater popularity, in part because actual results about number fields appear sooner in this treatment, in part simply because of the overpowering influence of Hilbert. His monumental *Zahlbericht*, the earliest extensive treatment of algebraic number theory as a subject in its own right, is highly ideal-theoretically oriented, and almost all later authors (for example, Hecke and Landau) have followed in Hilbert's tradition, a tradition largely responsible for the extraordinary development of abstract algebra, in particular ring and field theory, during the past half-century. But the pendulum has swung: most of the recent significant research in algebraic number theory, such as the Artin-Tate formulation of class field theory and Tate's work on zeta functions in number fields, or in applications to such related disciplines as the arithmetic theory of forms, is founded on the divisor-theoretic approach. Until quite recently, however, the only accessible sources treating number theory from the point of view of divisors have been Hasse's *Zahlentheorie* and the Artin N.Y.U.-Princeton notes, *Algebraic numbers and algebraic functions, I.* Neither of these is particularly suitable for someone wanting to *learn* the subject: Hasse's 601-page volume is a labyrinth to the uninitiated, moreover managing to miss the essentially *algebraic* (or "modern," if one will) flavor of this part of number theory, whereas the Artin notes are devoted almost completely to the local theory. There had thus lately arisen a need for a new exposition, concise as well as comprehensive, of the divisor-theoretic approach to the subject, or preferably, one which compares the two approaches and chooses the best features of both.

Edwin Weiss apparently recognized this need, for he has filled it admirably with his *Algebraic number theory*. Within considerably less than half the amount of space used by Hasse (and in a framework so up-to-date as to utilize commutative diagrams and exact sequences!), this book brings the reader to the point at which he can understand the current literature or any of the more specialized books in the field. Comprehensive it certainly is, as is evident from the chapter headings: after two introductory chapters on rank 1 valuations (the most convenient approach to prime divisors), there follows a chapter on local theory containing almost all of the non-class-field-theoretic material on local fields appearing in the aforementioned Artin notes; an elegant axiomatic treatment of multiplicative ideal theory, linking the classical arithmetic theory with the more abstract idea of a Dedekind domain (such as found in Zariski and Samuel, *Commutative algebra*); derivation of the "big" theorems of algebraic number theory, for example the Dirichlet unit theorem, using the powerful product formula for valuations, in the setting of idèles; and two chapters on applications to the simplest cases, namely, quadratic and cyclotomic fields. Interspersed throughout the book, and particularly in the final two chapters, are many interesting examples from elementary number theory; this book, indeed, is convincing evidence for the view that the latter subject is most meaningful when considered as a special case of the more general algebraic number theory, notwithstanding its inherent appeal for gifted high-school seniors and education majors.

Weiss has definitely written with the student in mind, for there are many paragraphs of motivation and explanation that are not essential to the strict logical development, and that might ordinarily (but, alas, not frequently enough) be supplied by a skilled lecturer. His treatment of the valuation-extension problem in the noncomplete case, one of the stickiest points in the divisor-theoretic approach, is exceptionally clear, thanks to his recognition—in the best tradition of the contemporary homologist (despite our facetious comment above regarding exact sequences)—of the value of preserving inclusion maps, rather than immediately identifying subsets as such. He correctly realizes the analogy between function fields and number fields, carrying it through much of the book; this necessitates the use (pp. 57–58) of certain formulas for norm, trace, and field polynomial for inseparable extensions, not particularly standard fare in the "full year van der Waerden type course" prescribed in the preface as prerequisite. One of the very few printed sources for this material, the interested reader may wish to note, is Chapter II of Zariski and Samuel (*op. cit.*).

On the matter of prerequisites, we must take issue with Weiss as to the amount of general topology he requires. Realizing that some facts concerning metric space convergence, analogous to properties of the real numbers, are necessary for the study of valuations, we also point out that the would-be reader of the book is likely to have studied at most the usual one-semester's worth of baby topology, and thus never have heard of uniformities, nor had much experience with topological groups—yet both these concepts appear early. The use of the latter can easily be avoided; the reader need only have a copy of Hasse, or of O'Meara's *Introduction to quadratic forms*, handy for reorientation along more prosaic lines. Far more serious, we feel, is the use of uniform topologies throughout the opening seven pages; and we wonder how many timid graduate students, potential number theorists, have been frightened away from the book, and perhaps the subject itself, because of this. After page 7, uniformities are never used again, and for the simple reason that they are unnecessary, in that the topology induced by a valuation turns out of course to be metric. The situation is as follows. A valuation ϕ on a field F is usually assumed to satisfy the *triangle inequality*, i.e., $\phi(a+b)\leqq\phi(a)+\phi(b)$, whereupon the function $d(a, b)=\phi(a-b)$ defines a metric on F; and *equivalent* valuations ϕ_1 and ϕ_2 (by definition, ϕ_1 a positive real power of ϕ_2) are easily seen to define the same topology. Now, for the product formula, it is necessary in certain cases to form a power of an honest-to-goodness valuation, resulting in a function which no longer satisfies the triangle inequality, and therefore cannot rightly be called a "valuation." Weiss, following the Artin notes, replaces the triangle inequality by the weaker axiom that there exist a real constant C with the property that $\phi(a)\leqq 1$ for a in F implies $\phi(1+a)\leqq C$, defines his uniform topology in terms of spherical neighborhoods $\{x \mid \phi(x-a)<\epsilon\}$, and only later observes that the triangle inequality holds if and only if C can be taken to be $\leqq 2$. But there are two ways around this. One is to require only the triangle axiom, and to realize, when working with the product formula, that the valuations under consideration may be deficient in one property; this is done successfully by O'Meara (*op. cit.*, cf. especially pp. 19, 65–67). And if this solution is unsatisfying, one can easily postpone topological considerations until after proving that part of Weiss' Theorem 1-1-4 which states that two valuations are equivalent if and only if, for all a in F, $\phi_1(a)<1$ implies $\phi_2(a)<1$; it is then easy to see that any valuation is equivalent to a valuation whose C is $\leqq 2$; only at this point should the topology, which will be metric, be brought in. These remarks should be sufficient indication of the way to avoid uniform topologies; and we strongly urge the author to consider revision of this material in any possible future edition, so as to bring the opening sections in line with the otherwise excellent organization of his subject matter.

One of the most attractive features of the book is the selection of exercises, ranging from simple numerical calculations to theorems whose proofs take several pages when written out in Hasse. The dedication page contains a quotation from Pirke Avoth, which may be loosely translated as "The basic thing is not studying but doing," and which may have been intended as an exhortation to the reader to work through these exercises; anyone having done so will emerge with considerable understanding of number theory. Again, we recommend keeping a copy of Hasse handy, for the purpose of cheating on some of the harder problems.

Despite the compactness of the book, few of the proofs require much between-the-lines elaboration. Errors, mostly typographical, are average for a first printing; much can be learned by hunting for and correcting them, though we are not recommending this as a substitute for careful proofreading.

It is questionable whether, as indicated in the preface, the entire book can be covered in a one-semester graduate-level course; more reasonable would be a two-semester course, beginning with Galois theory (especially inseparability) and perhaps some topology of metric spaces. A watered-down version of such a course could also fit nicely into the undergraduate curriculum, following the standard introductory year of Modern Algebra. Another possibility, at the graduate level, would be to put some class field theory at the end.

Mention of class field theory brings us to the following observation. Now that a reasonable account of modern algebraic number theory has appeared, the next gap to be closed is the lack of an expository treatment of the Artin-Tate class field theory. In particular, there is need for a rapid and direct exposition of the very specialized cohomology of finite groups used in Artin-Tate, a subject which presently can only be learned by struggling through one of the available treatises on homological algebra; Serre, in *Corps locaux*, develops many of the techniques peculiar to application to class field theory, but also refers at critical points to Cartan-Eilenberg or to Grothendieck's Tôhoku paper. In *Algebraic number theory*, Edwin Weiss has amply indicated his gifts of exposition and his concern for intelligibility, and we would most respectfully suggest that he consider writing a sequel.

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