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PERSISTENT AND INVARIANT FORMULAS RELATIVE TO THEORIES OF HIGHER ORDER

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Background. In first order model theory [8] a formula ϕ is said to be $\mathfrak{A}$-*persistent* (for extensions) if for any two models $\mathfrak{M}$, $\mathfrak{M}'$ of the set of axioms $\mathfrak{A}$, when $\mathfrak{M}'$ is an extension of $\mathfrak{M}$, all (sequences of) elements of (the base set of) $\mathfrak{M}$ which satisfy ϕ in $\mathfrak{M}$ also satisfy ϕ in $\mathfrak{M}'$. ϕ is $\mathfrak{A}$-persistent for restrictions if $\sim\phi$ is $\mathfrak{A}$-persistent for extensions; ϕ is $\mathfrak{A}$-*invariant* if both ϕ and $\sim\phi$ are $\mathfrak{A}$-persistent. The results which syntactically characterize the persistent and invariant formulas are as follows [8]: ϕ is $\mathfrak{A}$-persistent iff there is a purely existential ψ such that $\mathfrak{A}\vdash(\phi\leftrightarrow\psi)$, and ϕ is $\mathfrak{A}$-invariant iff there are purely existential (prenex) ψ and universal χ with $\mathfrak{A}\vdash(\phi\leftrightarrow\psi)\wedge(\psi\leftrightarrow\chi)$. Under suitable conditions on $\mathfrak{A}$, the latter result can be strengthened to give $\mathfrak{A}>(\phi\leftrightarrow\theta)$ for some quantifier-free θ.

1. **Introduction.** In higher order model theory the notion of *extension* has to be modified, as seen by considering well-founded models $\mathfrak{M}=\langle M, E\rangle$ of (fragments of) set theory. Assuming the axiom of extensionality, every such model is isomorphic to one $\mathfrak{M}^*=\langle M^*, E^*\rangle$ in which M^* is a transitive collection of sets and $E^*=\in\restriction M^*$. Given $\mathfrak{M}'=\langle M', E'\rangle$, an extension of $\mathfrak{M}$ in the first order sense, we have $M^*\subseteq(M')^*$ iff $(\forall a)_{\in M}(\forall b)_{\in M'}$ ($bE'a$ implies $b\in M$). This is equivalent to a certain relation $\mathfrak{M}\leqq\mathfrak{M}'$, which when examined provides a new notion of extension independent of the hypothesis of well-foundedness; the general definition of $\leqq$ is given in §3 below. The main problem is to find syntactic characterizations of the formulas which are persistent, resp. invariant for the new notion of extension. We give complete solutions to these and related problems both for

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ordinary logic and for ω-logic, superseding earlier fragmentary results [4], [7]. The proofs use some new generalizations of the interpolation theorem [1] given in [2]. All details of proofs will be presented elsewhere.

2. **Syntactic notions.** We consider the languages $\mathfrak{L}_\kappa = \mathfrak{L}_{\kappa,\omega}$ of [3] with or without identity, with at least one binary relation symbol $x\epsilon y$. Individual constant symbols are permitted but, for simplicity, no function symbols. We are particularly interested in the cases $\kappa=\omega$ and $\kappa=\omega_1$ (the first uncountable ordinal), i.e. in which the conjunctions $\prod$ and disjunctions $\sum$ permitted are always finite, resp. denumerable. However, we consider only formulas defining relations with a finite number of arguments; so all formulas will have only a finite number of free variables, and only finite strings of quantifiers $\bigwedge$, $\bigvee$ in front of any propositional connective. We shall also be interested in the case where the variables and individual constants are divided into different *sorts* or *types* in the sense of *many-sorted* logic. The basic logical rules are the same for these. In all cases and for any choice of κ and set $\mathfrak{A}$ of sentences of $\mathfrak{L}_\kappa$ we write $\mathfrak{A}\vdash\phi$ if ϕ is a consequence of $\mathfrak{A}$.

An occurrence of the quantifier $\bigvee y$ in a subformula $\bigvee y\psi$ of ϕ is said to be *restricted* to t if ψ is $(y\epsilon t\wedge\psi_1)$ where t is a term distinct from y. Similarly, in an occurrence $\bigwedge y\psi$ the quantifier is restricted to t if ψ is $(y\epsilon t\rightarrow\psi_1)$. A formula ϕ is said to be *completely restricted* if each quantifier occurrence is restricted; in this case for each quantified variable y there is a chain of quantified variable occurrences x_0, x_1, $\cdots$, x_n with $y=x_0$, each x_{i-1} restricted to x_i and x_n restricted to a free variable of ϕ or to a constant. A formula ϕ is said to be in (generalized) Σ-*form* if it is built up from atomic formulas and their negations using only conjunctions, disjunctions, existential quantifiers, and arbitrary restricted quantifiers. The dual notion is that of (generalized) Π-*form*, obtained by changing "existential" to "universal" here. ϕ is said to be in $\Pi\cap\Sigma$-*form relative to* $\mathfrak{A}$ if for some Π-formula ψ and Σ-formula χ we have $\mathfrak{A}\vdash(\phi\leftrightarrow\psi)\wedge(\psi\leftrightarrow\chi)$.

For $\kappa=\omega_1$ there is an important additional syntactic notion. The structure of any formula ϕ can be described by a certain countable tree T_ϕ, at each node of which is placed a number describing which operation is being applied to the predecessors. We say that ϕ has *hyperarithmetic structure* if T_ϕ is $\cong$ to a hyperarithmetic tree in the natural numbers. (It is not difficult to see that every such ϕ is also equivalent to a ψ for which T_ψ has a recursive isomorph, much as every hyperarithmetic well-ordering is isomorphic to a recursive one.)

Now, generalizing [3], one can also introduce languages $\mathfrak{L}^{(r)}_{\kappa,\omega}$ for infinite ordinals κ which are not cardinals, but admit a recursion theory; the formulae of $\mathfrak{L}^{(r)}_{\kappa,\omega}$ are required to be κ-finite (in the sense of meta-recursion theory [5]). If κ is the first nonrecursive ordinal, the formulae of $\mathfrak{L}^{(r)}_{\kappa,\omega}$ are then exactly those with hyperarithmetic structure.

3. **Model theoretic notions.** We consider structures $\mathfrak{M} = \langle M, \cdots \rangle$ associated with a given $\mathfrak{L}_\kappa$ and write $\epsilon_{\mathfrak{M}}$ for the binary relation corresponding to the symbol ϵ. If $a \in M$ we denote by $\epsilon_{\mathfrak{M}}$-$\mathrm{Cl}(a)$ *the transitive* $\epsilon_{\mathfrak{M}}$*-closure of* a, i.e. the smallest set S such that $a \in S$ and $(\forall b)_{\in S}[c \epsilon_{\mathfrak{M}} b$ implies $c \in S]$. We say that *ϵ-closure is definable in* $\mathfrak{A}$ if there is a formula $\chi(x, y)$ such that for any model $\mathfrak{M}$ of $\mathfrak{A}$ and $a, b \in M$, a, b satisfy χ iff $b \in \epsilon_{\mathfrak{M}}$-$\mathrm{Cl}(a)$. It is easily checked that ϵ-closure is definable in the simple theory of types and in set theory with the ω-rule.

To formulate the notion of extension mentioned in §1 we say that C is a *correspondence from* $\mathfrak{M}$ into $\mathfrak{M}'$, and write $C: \mathfrak{M} \leqq \mathfrak{M}'$, if C is a subset of $M \times M'$ for which the following conditions hold: (1) $(\forall a)_{\in M}(\exists a')_{\in M'} C(a, a')$; (2) for each pair of corresponding (say m-ary) relations $R_{\mathfrak{M}}$ and $R_{\mathfrak{M}'}$ if $C(a_i, a_i')$ for $i = 1, \cdots, m$ then $R_{\mathfrak{M}}(a_1, \cdots, a_m)$ iff $R_{\mathfrak{M}'}(a_1', \cdots, a_m')$; and (3) if $C(a, a')$ and $b' \epsilon_{\mathfrak{M}'} a'$ then $(\exists b)_{\in M} \cdot C(b, b')$. We write $\mathfrak{M} \leqq \mathfrak{M}'$ if there is such a C. This reduces to the notion suggested in §1 when $M \subseteq M'$ and C is the identity relation Id_M on M. If $=$ is one of the basic logical symbols of $\mathfrak{L}_\kappa$, i.e. if we consider only those structures in which $=_{\mathfrak{M}}$ is the identity relation on M, then $\mathfrak{M} \leqq \mathfrak{M}'$ iff $\mathfrak{M} \cong \mathfrak{M}_1$ where $M_1 \subseteq M'$ and $\mathrm{Id}_M: \mathfrak{M}_1 \leqq \mathfrak{M}'$.

Given $\mathfrak{M}$, $\mathfrak{M}'$, $a \in M$, $a' \in M'$ and $C \subseteq M \times M'$, we say that a, a' *are corresponding elements under* C and write $a \equiv_C a'$ if $C: \epsilon_{\mathfrak{M}}$-$\mathrm{Cl}(a) \leqq \epsilon_{\mathfrak{M}'}$-$\mathrm{Cl}(a')$, $C^{-1}: \epsilon_{\mathfrak{M}'}$-$\mathrm{Cl}(a') \leqq \epsilon_{\mathfrak{M}}$-$\mathrm{Cl}(a)$ and $C(a, a')$. If $=$ is one of the basic logical symbols of $\mathfrak{L}_\kappa$ and M, M' are transitive collections of sets with $\epsilon_{\mathfrak{M}} = \in \restriction M$, $\epsilon_{\mathfrak{M}'} = \in \restriction M'$ then these conditions hold iff $a = a'$ and hence $a \in M \cap M'$. More generally, this notion can be applied usefully whenever $\mathfrak{A}$ contains the axiom of extensionality and we are considering just those $\mathfrak{M}$ and $a \in M$ for which $\epsilon_{\mathfrak{M}}$ is well-founded on $\epsilon_{\mathfrak{M}}$-$\mathrm{Cl}(a)$.

If ϕ is a formula and $a = \langle \cdots a_i \cdots \rangle$ is an assignment to the free variables of ϕ which satisfies ϕ in $\mathfrak{M}$, we write $\mathfrak{M} \models \phi[a]$. We say that ϕ is $\leqq$*-persistent relative to* $\mathfrak{A}$ if whenever (i) $\mathfrak{M}$, $\mathfrak{M}'$ are models of $\mathfrak{A}$, (ii) a, a' are assignments in $\mathfrak{M}$, $\mathfrak{M}'$, resp., and (iii) there is $C: \mathfrak{M} \leqq \mathfrak{M}'$ with $C(a_i, a_i')$ for each pair of corresponding terms of a, a', then $\mathfrak{M} \models \phi[a]$ implies $\mathfrak{M}' \models \phi[a']$. We say that ϕ is $\leqq$*-invariant relative to* $\mathfrak{A}$ if both ϕ and $\sim\phi$ are $\leqq$-persistent relative to $\mathfrak{A}$. We say that ϕ is $\cap$-*invariant relative to* $\mathfrak{A}$ if whenever (i) $\mathfrak{M}$, $\mathfrak{M}'$ are models of $\mathfrak{A}$, (ii) a, a' are assignments in $\mathfrak{M}$, $\mathfrak{M}'$, resp., and (iii) there is a correspondence $C(\subseteq M\times M')$ with $a_i \equiv_C a_i'$ for each pair of corresponding terms of a, a' then $\mathfrak{M} \models \phi[a]$ iff $\mathfrak{M}' \models \phi[a']$. (These notions are to apply to any fixed choice of $\mathfrak{L}_\kappa$, including the many-sorted case.)

It is easy to check that if ϕ is $\cap$-invariant then ϕ is $\leqq$-invariant (rel. to any given $\mathfrak{A}$). Simple examples show that the converse is not in general true. There are special circumstances under which it is true, e.g. if all models of $\mathfrak{A}$ are well-founded and extensional, and the intersection of any two transitive models of $\mathfrak{A}$ is again a model of $\mathfrak{A}$. The first two of these hypotheses can be assured in theories of transfinite types of not too high rank (by means of axioms in $\mathfrak{L}_\kappa$ for suitable κ).

4. **Principal results.** We assume throughout the remainder of this paper that $\kappa=\omega$ or $\kappa=\omega_1$. Further, $\mathfrak{A}$ *is assumed to be denumerable in the case* $\kappa=\omega_1$; it is not essential to assume this in the case $\kappa=\omega$.

THEOREM 1. *ϕ is $\leqq$-persistent rel. to $\mathfrak{A}$ iff there is a Σ-formula ψ such that* $\mathfrak{A}\vdash(\phi\leftrightarrow\psi)$.

THEOREM 2. *If ϵ-closure is definable in $\mathfrak{A}$ then ϕ is $\cap$-invariant rel. to $\mathfrak{A}$ iff there is a completely restricted formula ψ such that* $\mathfrak{A}\vdash(\phi\leftrightarrow\psi)$.

THEOREM 3. *In the case $\kappa=\omega_1$, if $\mathfrak{A}\cup\{\phi\}$ is a Π_1^1 set of formulas with hyperarithmetic structure then the formulas ψ in Theorems* 1 *and* 2 *can also be chosen to have hyperarithmetic* (*or even recursive*) *structure.*

To obtain results for ω-models one considers $\mathfrak{L}_{\omega_1}$ logic with $=$ with at least a 0-sort of variable $x^0, \cdots$, individual constant symbols $\bar{n}$, for $n<\omega$, and $\bar{\omega}$, and axioms $\bigwedge x^0[x^0\epsilon\bar{\omega}\leftrightarrow\sum_{n<\omega}(x^0=\bar{n})]$ and $\prod_{n\neq m}(\bar{n}\neq\bar{m})$.

5. **Methods of proof.** The principal method consists in using appropriate generalizations of Craig's interpolation theorem [**1**]. These are the following.

THEOREM 4 [2]. *Suppose $\vdash(\phi\rightarrow\theta)$ in many-sorted logic, where ϕ, θ are formulas with common symbols and sorts of variables. Then there is a formula ψ which involves only these common symbols and sorts for which* $\vdash(\phi\rightarrow\psi)\wedge(\psi\rightarrow\theta)$.

It does not matter here whether we are considering logic without or with $=$. If we allow $=$ only between objects of the same sort, it is a direct matter to reduce the problem to the former case. K. Kunen has shown that it is still possible to carry out such a reduction even if we allow $=$ between objects of arbitrary sort. Call ψ *an interpolating formula for* ϕ, θ if it satisfies the conclusion of Theorem 4. In the case of $\mathcal{L}_{\omega_1}$, Theorem 4 also generalizes [6]. One has further:

THEOREM 5 [2]. *In* $\mathcal{L}_{\omega_1}$, *if* ϕ, θ *both have hyperarithmetic structure and satisfy the hypotheses of Theorem* 4 *then we can find an interpolating formula* ψ *for* ϕ, θ *which has hyperarithmetic* (*or even recursive*) *structure.*

Theorem 2 follows from Theorem 4 and Theorem 3 makes additional use of Theorem 5. To prove Theorem 1 in the case of $\mathcal{L}_\omega$, we use Henkin's method of constants applied to a many-sorted language to reduce to a propositional interpolation theorem. In the case of $\mathcal{L}_{\omega_1}$ (to get first the corresponding result for persistence under $\leqq$-restrictions) we generalize arguments communicated to us by J. I. Malitz, giving an interpolation theorem for Π-formulas.

Theorem 4 for $\mathcal{L}_\omega$ with $=$ between arbitrary sorts can also be used to give simple new proofs of definability results in first order model theory, because the many-sorted calculus allows one to express syntactically and simply that one model is an extension of another. Let $\mathfrak{A}$ and $\mathfrak{A}_1$ be two sets of sentences. A formula ϕ is called $(\mathfrak{A}, \mathfrak{A}_1)$-*invariant* if for any model $\mathfrak{M}$ of $\mathfrak{A}$, assignment a in $\mathfrak{M}$, and extensions $\mathfrak{M}'$, $\mathfrak{M}''$ of $\mathfrak{M}$ which are models of $\mathfrak{A}_1$, we have $\mathfrak{M}' \models \phi[a]$ iff $\mathfrak{M}'' \models \phi[a]$. (If $\mathfrak{A}$ is such that every model $\mathfrak{M}$ of $\mathfrak{A}$ can be extended to a model of $\mathfrak{A}_1$, this notion reduces to the following notion of [8]: $\mathfrak{A}_1$ is model-consistent rel. to $\mathfrak{A}$ and ϕ is invariant rel. to $\mathfrak{A}_1$ over $\mathfrak{A}$.) Consider three homologous languages $\mathcal{L}$, $\mathcal{L}'$, $\mathcal{L}''$ obtained by using three sorts of variables $x, y, \cdots, x', y', \cdots$, and $x'', y'', \cdots$, resp. (The dashes will be used in an obvious way to indicate the homologues of *bound* variables and of relation symbols.) Let Ext $(\mathcal{L}, \mathcal{L}')$ be the set of sentences $\bigwedge x \bigvee x'\ (x=x')$ and for each relation symbol R of $\mathcal{L}$, $\bigwedge x, y, \cdots [R(x, y, \cdots) \leftrightarrow R'(x, y, \cdots)]$. Then $(\mathfrak{M}, \mathfrak{M}')$ is a model of $\mathrm{Ext}(\mathcal{L}, \mathcal{L}')$ iff $\mathfrak{M}'$ is an extension of $\mathfrak{M}$. Thus under the above hypotheses $\mathfrak{A} \cup \mathfrak{A}_1' \cup \mathfrak{A}_1'' \cup \mathrm{Ext}(\mathcal{L}, \mathcal{L}') \cup \mathrm{Ext}(\mathcal{L}, \mathcal{L}'') \vdash (\phi' \leftrightarrow \phi'')$. It follows from Theorem 4 that there is a ψ in the language $\mathcal{L}$ such that $\mathfrak{A} \cup \mathfrak{A}_1' \cup \mathrm{Ext}(\mathcal{L}, \mathcal{L}') \vdash (\phi' \rightarrow \psi)$ and $\mathfrak{A} \cup \mathfrak{A}_1'' \cup \mathrm{Ext}(\mathcal{L}, \mathcal{L}'') \vdash (\psi \rightarrow \phi'')$. Hence by identifying $\mathcal{L}'$ and $\mathcal{L}''$ we strengthen the former to proving $(\phi' \leftrightarrow \psi)$. By the above this implies Theorem 5.3.1 of [8].

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