

# WIENER-HOPF TYPE PROBLEMS FOR ELLIPTIC SYSTEMS OF SINGULAR INTEGRAL EQUATIONS<sup>1</sup>

BY ELIAHU SHAMIR

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The problem treated in this paper is roughly the inversion of elliptic systems of singular integral equations in a half-space of  $R^n$ . Ellipticity means that the system is invertible over the whole of  $R^n$ , in our case explicitly. We first introduce notation and some spaces of (vector-valued) distributions.

Let  $(x, y)$  denote points in  $R^n$  with  $x \in R^{n-1}$ ,  $y \in R$ .  $R_+^n [R_-^n]$  is the half-space  $y \geq 0$  [ $y \leq 0$ ].  $H^{s,p}$  is the space of distributions  $u$  for which

$$\|u\|_{s,p} = \|F^{-1}(1 + |\xi|^2 + \eta^2)^{s/2}Fu\|_{L^p} < \infty.$$

Here  $(Fu)(\xi, \eta) = \int u(x, y)e^{i(x \cdot \xi + y \cdot \eta)} dx dy$ , with  $(\xi, \eta)$  dual to  $(x, y)$ . We assume  $1 < p < \infty$ .  $H_+^{s,p}$  is the subspace of elements supported in  $R_+^n$ .  $H_+^{s,p}(R_+^n)$  is the quotient  $H_+^{s,p}/H_-^{s,p}$  (it is a space of distributions on  $\dot{R}_+^n$ , the open half-space).  $Y_+$  denotes the canonical map onto the quotient and  $\|Y_+u\|_{s,p}$  is the quotient norm.  $H_+^{s,p}$ ,  $H_+^{s,p}(R_+^n)$  and  $Y_+$  are similarly defined. For  $s=0$ , we can identify  $H_{\pm}^{0,p} = L_{\pm}^p$  with  $L^p(R_{\pm}^n)$ , and  $Y_{\pm}$  with multiplication by the characteristic function of  $R_{\pm}^n$ . The definitions above extend to vector valued functions component-wise.

Let  $M(\xi, \eta)$  be an  $N \times N$  matrix of functions, positively homogeneous of degree 0,  $C^{l+1}$  on  $|\xi|^2 + \eta^2 = 1$  where  $l > n/2$ . The operator  $\mathbf{M} = F^{-1}M(\xi, \eta)F$  (whose symbol is  $M(\xi, \eta)$ ) is bounded in  $H^{s,p}$ , invertible (elliptic) if  $\det [M(\xi, \eta)] \neq 0$  for  $(\xi, \eta) \neq 0$ .

**THEOREM A.** *The operator  $\tilde{\mathbf{M}}: u \rightarrow (Y_-u, Y_+Mu)$  has a closed range in  $H^{s,p}(R_-^n) \times H^{s,p}(R_+^n)$  for every  $s$  except at most  $N$  exceptional values of  $s \pmod{1}$ . There exists  $k' \geq k''$  such that for  $s = k + \sigma$  nonexceptional*

$$\begin{aligned} \|u\|_{s,p} &\leq C[\|Y_-u\|_{s,p} + \|Y_+Mu\|_{s,p}], \quad \text{all } u \in H^{s,p}; k \geq k', \\ \sum_{\pm} \|V_{\pm}\|_{-s,p'} &\leq C\|V_- + MV_+\|_{-s,p'}, \quad \text{all } V_{\pm} \in H_{\pm}^{-s,p'}; k \leq k''; \\ &k'' = k' \text{ in the scalar case.} \end{aligned}$$

The first estimate means that  $\tilde{\mathbf{M}}$  is 1-1 and has a closed range. The second ("dual") estimate assures that the range of  $\tilde{\mathbf{M}}$  is full. Thus as  $s \rightarrow +\infty$  the operator  $\tilde{\mathbf{M}}$  becomes left invertible, as  $s \rightarrow -\infty$  it becomes right invertible.

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We proved those estimates before for  $n=1$  and general  $M$  [6], or  $M=I$  (the identity matrix) and general  $n$  [7]. In the latter case the exceptional values are  $s=1/p \pmod{1}$ . (This is so whenever  $M(0, 1) = M(0, -1)$ .) In fact, Theorem A shows that  $\tilde{M}$  for general  $M$  behaves pretty much like  $\tilde{I}$ .

Theorem A follows from:

**THEOREM B.** *The operator  $Y_+u \rightarrow Y_+MY_+u$  has a closed range in  $L^p(\mathbb{R}_+^n)$  if and only if the eigenvalues of  $M(0, -1) \cdot M^{-1}(0, 1)$  do not lie on the ray  $\lambda = 2\pi/p$ . Let*

$$(1) \quad M_k(\xi, \eta) = (\eta - i|\xi|)^k M(\xi, \eta) (\eta + i|\xi|)^{-k}.$$

*There are integers  $k', k''$  such that  $Y_+M_{k'}Y_+$  is 1-1 if  $k \geq k'$ , onto if  $k \leq k''$  (provided  $Y_+MY_+$  has a closed range).*

**REMARK.** The estimates of Theorem A settle the problem of obtaining a priori  $L^p$  estimates for elliptic partial differential equations in  $n+1$  dimensional domains with piecewise smooth boundary operators (so called "mixed" problems). These (usually  $L^2$ ) estimates were obtained before under very special conditions [3], [4], [5] and [8].

**METHOD OF PROOF.** For Theorem B, we first prove the following reduction:

**LEMMA 1.** *Let  $M_{\tilde{\xi}} = F^{-1}M(\tilde{\xi}, \eta|\tilde{\xi}|^{-1})F$ . Then estimate*

$$\|Y_+u\| \leq C \|Y_+MY_+u\|, \quad u \in L^p$$

*(norms are  $L^p(\mathbb{R}_+^n)$  norms) is equivalent to the family of estimates*

$$\|Y_+u\| \leq C \|Y_+M_{\tilde{\xi}}Y_+u\|, \quad u \in L^p,$$

*for all  $\tilde{\xi}$  satisfying  $|\tilde{\xi}| = 1$ .*

The operators  $M_{\tilde{\xi}}$  are easier to study since their symbols are essentially one-dimensional. As usual with this type of problem, one tries to factor the symbol into product of matrices holomorphic in  $\text{Im } \eta > 0$  and  $\text{Im } \eta < 0$ . Here it suffices to factor  $M(\tilde{\xi}, \eta)$  for fixed  $\tilde{\xi}$  and then substitute  $\eta|\tilde{\xi}|^{-1}$  for  $\eta$ . We use results of Gohberg-Krein [1], [2]. They factor matrices of the form  $I+K(\eta)$  where  $K(\eta) \in FL^1$  (or a suitable subring, cf. [1]). Thus we first have to "fill in" the jump of  $M(\tilde{\xi}, \eta)$  at infinity in case  $M(0, 1) = M(\tilde{\xi}, \infty) \neq M(\tilde{\xi}, -\infty) = M(0, -1)$ . If the jump matrix

$$(2) \quad M(0, -1) \cdot M^{-1}(0, -1) \text{ is similar to } \text{diag} [\lambda_1, \dots, \lambda_N],$$

this is readily done by diagonal factors of the form

$$(3) \quad (\eta \pm i)^\sigma = \text{diag} [(\eta \pm i)^\sigma, \dots, (\eta \pm i)^\sigma]$$

where  $\sigma = (\sigma_1, \dots, \sigma_N)$  is  $N$ -tuple of fractional (may be complex) numbers determined by  $\lambda_1, \dots, \lambda_N$ . Indeed  $M_1(\xi, \eta) = (\eta - i)^{-\sigma} M(\xi, \eta)(\eta + i)^\sigma$  has the same value at  $\pm \infty$  and is factorizable. Factoring it, we get for  $M(\xi, \eta)$

LEMMA 2. For a fixed  $\xi \neq 0$  we have (suppressing the dependence on  $\xi$ ):

$$(4) \quad M(\xi, \eta) = Q_-^{-1}(\eta)(\eta - i)^\sigma \left( \frac{\eta - i}{\eta + i} \right)^\kappa (\eta + i)^{-\sigma} Q_+(\eta)$$

where  $Q_+(\eta)$  [ $Q_-(\eta)$ ] and its inverse are bounded and smooth for real  $\eta$ , have holomorphic extension to  $\text{Im } \eta > 0$  [ $\text{Im } \eta < 0$ ]. Moreover, their derivatives decrease as  $|\eta| \rightarrow \infty$  in a manner which assures that  $Q_\pm(\eta|\xi|^{-1})$  and their inverses are  $L^p$ -multipliers.  $\kappa = (\kappa_1, \dots, \kappa_N)$  is a nonincreasing sequence of integers which are uniformly bounded for  $|\xi| = 1$ .

REMARK. If (2) is not satisfied, the factorization of  $M$  is more complicated but the final results remain unchanged.

LEMMA 3. Let  $Q_\pm = F^{-1}Q_\pm(\eta|\xi|^{-1})F$  and

$$(5) \quad D_{\kappa+\sigma} = F^{-1}(\eta - i|\xi|)^\sigma \left( \frac{\eta - i|\xi|}{\eta + i|\xi|} \right)^\kappa (\eta + i|\xi|)^{-\sigma} F.$$

Then  $Q_+$  sets an isomorphism between the null-spaces of  $Y_+M_\xi Y_+$  and  $Y_+D_{\kappa+\sigma}Y_+$ .  $Q_-$  sets an isomorphism between the ranges of these operators in  $L^p(R_+^n)$ .

Notice that  $D_{\kappa+\sigma}$  is a direct sum of scalar operators for which we have

LEMMA 4. If  $k$  is an integer and  $-1 + 1/p < \text{Re } \sigma \leq 1/p$  then the (scalar) operator  $Y_+D_{\kappa+\sigma}Y_+$  has a closed range if and only if  $\text{Re } \sigma \neq 1/p$ . In this case it is 1-1 if  $k \geq 0$ , onto if  $k \leq 0$ .

The proof of Theorem B follows now quite easily from Lemmata 1-4; for Theorem A we need two more. Let us denote

$$J_\pm^t = F^{-1}[\eta \pm i(1 + |\xi|^2)^{1/2}]^t F.$$

LEMMA 5.  $J_-^t$  maps  $H^{s,p}(R_+^n)$  onto  $H^{s-t,p}(R_+^n)$  (in particular onto  $L^p(R_+^n)$  if  $s=t$ ) and  $\|Y_+J_-^t u\|_{s-t,p} \sim \|Y_+u\|_s$ . (Similar statement with  $+$  and  $-$  interchanged.)

LEMMA 6. The following estimates are equivalent

$$\begin{aligned} \|u\|_{s,p} &\leq C[\|Y_-u\|_{s,p} + \|Y_+Mu\|_{s,p}], \quad u \in H^{s,p}, \\ \|v\|_{0,p} &\leq C[\|Y_-v\|_{0,p} + \|Y_+J^sMJ_+^{-s}v\|_{0,p}], \quad v \in L^p, \\ \|Y_+v\|_{0,p} &\leq C\|Y_+M_sY_+v\|, \quad v \in L^p \quad (\text{cf. (1)}), \\ \sum_{\pm} \|V_{\pm}\|_{-s,p'} &\leq C\|V_- + MV_+\|_{-s,p'}, \quad V_{\pm} \in H^{-s,p'}. \end{aligned}$$

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NORTHWESTERN UNIVERSITY