Hence the $C^*$-algebra $\pi_x(A)$ and so $A$ have a type III-factor $*$-representation.

This completes the proof.

REFERENCES

1. J. W. Calkin, Two sided ideals and congruences in the ring of bounded operators in Hilbert space, Ann. of Math. 42 (1941), 839–873.

SOME UNSYMMETRIC COMBINATORIAL NUMBERS

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By an $n$-configuration we shall mean an abstract set of $n$ elements, together with the set of all unordered pairs of distinct elements from the set. It is convenient also to use quasi-geometrical terminology such as vertex for element, edge or side for a pair (2-tuple), triangle as well as triple (3-tuple) for a 3-subconfiguration, and so on.

The Ramsey number $N(p, q, 2)$ (see [3, pp. 38–43], or [2, pp. 61–65]), for two kinds $h, v$ of pairs (or two “colors of edges”), is the smallest integer such that if $n \geq N(p, q, 2)$, then any $n$-configuration is sure to contain either an $h$ $p$-tuple (a $p$-tuple all of whose edges are $h$) or a $v$ $q$-tuple. Call a $p$-tuple all of whose edges are alike ($h$ or $v$) a like $p$-tuple. We introduce, and partially determine the values of, new analogous combinatorial numbers $K(p, q, 2)$, $M(p, q, 2)$, and $V(p, q, 2)$.

DEFINITIONS. The number $K(p, q, 2)$ is the smallest integer such that if $n \geq K(p, q, 2)$, then for each vertex, the configuration is sure to contain either a like $p$-tuple containing the vertex, or a like $q$-tuple not containing the vertex. For three kinds $r, g, v$ of edges, $M(p, q, 2)$
is the smallest integer such that if \( n \geq M(p, q, 2) \), the configuration is sure to contain either a like \( p \)-tuple, or a \( j, k \)-tuple (a \( q \)-tuple having at most two kinds \( j \), \( k \) of edges, where \( j, k = r, g, \) or \( v \)). The number \( V(p, q, 2) \) is the smallest integer such that if \( n \geq V(p, q, 2) \), then for each vertex of the configuration, the configuration contains either a like \( p \)-tuple containing the vertex, or a \( j, k \)-tuple not containing the vertex.

Consider for a moment "vertical" numbers, which otherwise are like the Ramsey numbers: \( S(p, q, 2) \), for example, is the smallest integer such that a configuration with \( n \geq S(p, q, 2) \) is sure to contain, for each vertex, either an \( h \)-tuple containing the vertex, or a \( v \)-tuple not containing the vertex. Evidently \( N(p, q, 2) \leq S(p, q, 2) \). But for all \( p \geq 3 \), \( q \geq 3 \), \( S(p, q, 2) = \infty \): for arbitrarily large \( n \), at one vertex, assign \((p - 1)\) edges from the vertex to be \( h \), the remainder \( v \). Let one edge joining a pair of other ends of the \((p - 1)\) edges be \( v \), and let all other edges of the \( n \)-configuration be \( h \). Then for the vertex, the \( n \)-configuration contains neither an \( h \)-tuple containing the vertex, nor a \( v \)-tuple not containing the vertex. Moreover \( S(p, q, 2) = \infty \) for all \( p \geq 2 \), \( q \geq 2 \).

Denote by \( W(q, p, 2) \) the smallest integer such that if \( n \geq W(q, p, 2) \), then for each vertex the configuration is sure to contain either a \( j, k \)-tuple containing the vertex, or a like \( p \)-tuple not containing the vertex. We notice that \( W(q, p, 2) = V(p, q, 2) \).

Our results so far concerning the numbers \( K, M, V \) are indicated in the following Theorem 1 (including the table) and Theorem 2. For purposes of comparison, the known values of the Ramsey numbers \( N \) also are included: the entries in the table are the values of \( N, K, M, V \) in that order, for each \( p, q \).

**Theorem 1.** For \( p, q \) from 3 to 5 inclusive, the numbers have values as given in the following table (cf. the table in [3, p. 42]).

<table>
<thead>
<tr>
<th></th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6, 6, 5, 6</td>
<td>9, 8, 8, 10 or 11</td>
<td>14, 10, 14, 14</td>
</tr>
<tr>
<td>4</td>
<td>9, 7, 5, 6</td>
<td>18, 18, 10 to 17, 11 to 18</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>14, 7, 5, 6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For all \( p > 3 \), we have that \( K(p, 3, 2) = 7 \); for all \( q \geq 3 \), that \( K(q, q, 2) = N(q, q, 2) \) and \( K(3, q, 2) = 2q; \) for \( p > q \), that \( K(p, q, 2) \leq N(q, q, 2) + 1 \); and for \( q > p > 3 \), that \( K(p, q, 2) \leq \max(K(p - 1, q, 2) + 1, 2q + p - 3) \). Further, \( N(3, 6, 2) \) is 17 or 18 (cf. [1]), and 17 = \( M(3, 6, 2) \leq N(3, 6, 2) \).
A configuration is called degenerate, with respect to any of the combinatorial numbers, in case it does contain (for each vertex in case of \( V \)) either a \( p \)-tuple or a \( q \)-tuple as described in the corresponding definition. As an example, it is quite easy to find an 8-configuration which is nondegenerate with respect to \( N(3, 4, 2) \)—an octagon with an 8-cycle plus a 4-cross (see below) of blue edges, and an 8-cycle plus two 4-cycles of red edges, has neither a blue triangle nor a red quadruple (cf. the existence proof for the 8-configuration in [1]). The method of establishing the above lower bounds \( L \) is to exhibit in each case a nondegenerate configuration with \( n = L - 1 \). To establish an upper bound \( U \), it is sufficient to show that any configuration with \( n = U \) must be degenerate. The value of a combinatorial number of course is determined in case \( L = U \). Details will be included in a paper which will be offered for publication elsewhere.

A subsidiary result, analogous to Steiner triple systems ([2] or [3]), is the following. A \( k \)-cycle is a closed string of \( k \) successively adjacent edges, such as \( 12; 23; \cdots; k-1, k, 1 \), where \( \{1, \cdots, k\} \) is a subset of \( k \) of the vertices of the configuration. In any \((2n+1)\)-configuration, the edges can be covered (each exactly once) by \( n \) \( k \)-cycles with \( k = (2n+1) \). A \( k \)-cross is a set of \( k \) edges, no two of which are adjacent. The edges of any \((2n+2)\)-configuration can be covered by \( n \) \((2n+2)\)-cycles and an \((n+1)\)-cross.

**Theorem 2.** We have \( K(p, q, 2) \leq N(p, q, 2) \), \( M(p, q, 2) \leq N(p, q, 2) \), \( M(p, q, 2) \leq V(p, q, 2) \). For each \( q \), \( V(q, q, 2) \) is either \( M(q, q, 2) \) or \( M(q, q, 2) + 1 \); for any \( p \), \( V(p, q, 2) \leq M(q, q, 2) + 1 \). For \( p > q \), \( M(q, q, 2) \leq \min(M(p, q, 2), V(p, q, 2)) \), and \( V(q, q, 2) \leq V(p, q, 2) \). For \( p \geq 3 \), \( M(p, 3, 2) = 5 \), and \( V(p, 3, 2) = 6 \). For \( q > 3 \), \( V(3, q, 2) \leq (3q - 1) \).

Reference [4], for example, indicates the wealth of possible applications for combinatorial results.

**References**


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