HILBERT SPACE IS HOMEOMORPHIC TO THE COUNTABLE INFINITE PRODUCT OF LINES[1]

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1. Introduction. In this paper, Hilbert space, denoted by l_2, is understood to be the space of all sequences (x_i) such that $\sum_{i=1}^{\infty} x_i^2 < \infty$ with $d((x_i), (x_i')) = (\sum_{i=1}^{\infty} (x_i - x_i')^2)^{1/2}$. We let the countable infinite product of lines be regarded as $s = \prod_{i=1}^{\infty} I_i^0$ where, for each $i > 0$, I_i^0 denotes the open interval $(0, 1)$.

Let the symbol "$\sim$" mean "is homeomorphic to." We shall prove

THEOREM I. $l_2 \sim s$.

As a consequence of this theorem it is possible to investigate topological properties of l_2 as topological properties of s. In turn s is a "natural" subset of the Hilbert cube (the countable infinite product of closed intervals) which facilitates the study of s.

In 1928 in [**5**, pp. 94–96] Fréchet raised the general question as to which linear topological spaces were homeomorphic to each other. Specifically he asked whether l_2 (called Ω) was homeomorphic to s (called E_ω).

In 1932 in [**2**, p. 233], Banach stated that Mazur had shown that s was not homeomorphic to l_2. Subsequently it was understood that the question was still open.

The topological classification of complete linear metric spaces initiated by Fréchet has been the subject of considerable research activity with noteworthy contributions by Bessaga, Kadec, Klee and Pełczyński among others. See the bibliography in [**3**]. Particular attention has been given to Fréchet spaces: locally convex complete linear metric spaces. With Theorem I of this paper and recent profound results of Kadec and of Bessaga and Pełczyński, the topological classification of separable infinite-dimensional Fréchet spaces is now complete. All such spaces are homeomorphic to each other.

The results leading to this theorem are the following. In a paper to be published in Dokl. Akad. Nauk SSSR, Kadec gives a proof of the theorem "All separable infinite-dimensional Banach Spaces are homeomorphic." Earlier in [**4**] and in [**3**, Theorem 9.2], Bessaga and Pełczyński have shown "Under the conjecture that all separable in-

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finite-dimensional Banach spaces are homeomorphic with l_2, every separable infinite-dimensional Fréchet space X, with $X \neq s$, is homeomorphic with l_2." Therefore, with Theorem I of this paper, the classification is complete.

2. **The strategy of the proof of Theorem** I. The proof of Theorem I uses only standard topological methods and two recent results, (A) and (B) below, which are not proved here. (However, an outline of the proof of (B) is given in §3.) Otherwise, this paper is self-contained.

(A) (Bessaga-Pełczyński) [**3**] or [**4**].

$$l_2 \sim l_2 \times s.$$

(B) [**1**]. For any separable metric space Z and any countable collection $\{K_i\}_{i\geq 1}$ of compact subsets of $Z \times s$,

$$\left[(Z \times s) \Big\backslash \bigcup_{i=1}^{\infty} K_i\right] \sim Z \times s.$$

Taking Z as a single-point set, the following theorem is a corollary of (B).

(C) For any countable collection $\{K_i\}_{i\geq 1}$ of compact subsets of s,

$$\left[s \Big\backslash \bigcup_{i=1}^{\infty} K_i\right] \sim s.$$

Taking $Z \sim l_2$, the following theorem is a corollary of (A) and (B).

(D) For any countable collection $\{K_i\}_{i\geq 1}$ of compact subsets of l_2,

$$\left[l_2 \Big\backslash \bigcup_{i=1}^{\infty} K_i\right] \sim l_2.$$

With these observations, the strategy of the proof is the following. We exhibit, in §4, a particular set $\bar{l}_2$ which, by (D), we show to be homeomorphic to l_2. Then, in §5, we show (E) a homeomorphism f of $\bar{l}_2$ into s such that $s \backslash f(\bar{l}_2)$ is the countable union of compact sets. By (C), $f(\bar{l}_2) \sim s$. Thus we have

$$l_2 \overset{\mathrm{D}}{\sim} \bar{l}_2 \overset{\mathrm{E}}{\sim} f(\bar{l}_2) \overset{\mathrm{C}}{\sim} s$$

and hence $l_2 \sim s$.

3. **Outline of a proof of** (B). We have $s = \prod_{j=1}^{\infty} I_j^0$ where, for each j, I_j^0 is the open interval (0, 1). Then we may write s as $\pi_{i=1}^{\infty} s_i$ where, for each i, s_i is a countable infinite product of factors I_j^0. Regarding $Z \times s$ as $Z \times \pi_{i=1}^{\infty} s_i$, we let g_i be the projection of $Z \times s$ onto s_i. Observe

that, for each i with K_i as in the statement of (B), $g_i(K_i)$ is compact.

It can be shown (as in §3 of [1]) that, for each i, there is a homeomorphism ρ_i of s_i onto itself and a factorization $s_i = s_i' \times s_i''$ (where each of s_i' and s_i'' is a countable infinite product of factors I_j^0) such that if σ_i denotes the projection of $s_i' \times s_i''$ onto s_i', then $\sigma_i \rho_i g_i(K_i)$ is a single point. In other words, $\rho_i g_i(K_i)$ has infinite deficiency in s_i.

Let ρ be the homeomorphism of $Z \times \prod_{i=1}^{\infty} s_i$ onto itself defined coordinatewise as ρ_i on s_i and as the identity on Z. It now suffices to exhibit a homeomorphism of $(Z \times s) \setminus \bigcup_{i=1}^{\infty} \rho(K_i)$ onto $Z \times s$.

In what follows we let $K_0 = \rho(K_0)$ be the null set.

Considering $Z \times s$ as $Z \times \prod_{i=1}^{\infty} (s_i' \times s_i'')$, we may exhibit (as in §5 of [1]) a sequence $\{h_i\}_{i \geq 1}$ such that

(1) for each $i \geqq 1$, h_i is a homeomorphism of $(Z \times s) \setminus \bigcup_{j=0}^{i} \rho(K_j)$ onto $(Z \times s) \setminus \bigcup_{j=0}^{i-1} \rho(K_j)$,

(2) for each $i \geqq 1$, h_i affects only coordinates in s_i', and

(3) the infinite composition $\cdots \cdot h_3 \cdot h_2 \cdot h_1$ is a homeomorphism h carrying $(Z \times s) \setminus \bigcup_{j=0}^{\infty} \rho(K_j)$ onto $Z \times s$.

Thus $h\rho$ will be the desired homeomorphism of $(Z \times s) \setminus \bigcup_{j=0}^{\infty} K_j$ onto $Z \times s$.

In conclusion, we remark that the homeomorphism h_i can be considered as a modification of a homeomorphism defined coordinatewise as (1) the identity on coordinate spaces other than s_i' and (2) a homeomorphism which moves the single point $\sigma_i g_i \rho(K_i)$ off s_i'. The modification requires that h_i move only $\rho(K_i) \setminus \bigcup_{j=0}^{i-1} \rho(K_j)$ off $Z \times s$ rather than the larger set $g_i^{-1} \sigma_i^{-1}(\sigma_i g_i \rho(K_i)) \setminus \bigcup_{j=0}^{i-1} \rho(K_j)$.

4. **Description of l_2.** For each $i \geqq 1$, let $W_i = \{(x_j) \mid (x_j) \in l_2$ and for all $k > i$, $x_k = 0\}$. Then for each i, W_i is homeomorphic to the Euclidean space E_i and is the countable union of compact sets. Thus $W = \bigcup_{i=1}^{\infty} W_i$ is the countable union of the compact sets. Let $M_1 = \{(x_j) \mid (x_j) \in l_2 \setminus W$ and $x_1 = 1\}$. Let $\bar{l}_2 = \{(x_j) \mid (x_j) \in l_2 \setminus W$ and $\sum_{j=1}^{\infty} x_j^2 = 1\}$. We observe that $\bar{l}_2 \sim M_1 \sim (l_2 \setminus W) \sim l_2$. The first homeomorphism follows by projection from the point (x_i) for which $x_1 = -1$ and $x_j = 0$ for $j > 1$. The second homeomorphism follows by the formula $(1, x_2, x_3, \cdots) \to (x_2, x_3, x_4, \cdots)$ and the third by Statement (D) of §2. Thus $\bar{l}_2$ is homeomorphic to l_2.

We note, for use in §5, that

$$\bar{l}_2 = \left\{ (x_j) \mid (x_j) \in l_2, \ \sum_{j=1}^{\infty} x_j^2 = 1 \quad \text{and,} \quad \text{for each } i \geqq 1, \ \sum_{j=1}^{i} x_j^2 < 1 \right\}.$$

5. **Description of the homeomorphism f of $\bar{l}_2$ into s.** With (x_i) denoting a point of $\bar{l}_2$ and (y_i) denoting a point of s (where $0 < y_i < 1$,

for each i) we define a function f coordinatewise as:

$$y_1 = (1 + x_1)/2 \quad \text{and} \quad \text{for each } i > 1,$$

$$y_i = \frac{\left(1 - \sum_{j=1}^{i-1} x_j^2\right)^{1/2} + x_i}{2\left(1 - \sum_{j=1}^{i-1} x_j^2\right)^{1/2}}.$$

The verification of the following four properties will complete the proof of our Theorem I:

(1) f is one-to-one from $\bar{l}_2$ into s,
(2) f is continuous,
(3) f^{-1} is continuous,
(4) $s\backslash f(\bar{l}_2)$ is the countable union of compact sets.

(1) For any $i>0$ and any point $(x_i)\in\bar{l}_2$, we have $\sum_{j=1}^{i} x_j^2<1$. Therefore $|x_i|<(1-\sum_{j=1}^{i-1} x_j^2)^{1/2}$ and thus $0<y_i<1$. Hence f carries $\bar{l}_2$ into s. Also if (x_i) and (x_i') are different points of $\bar{l}_2$ with images (y_i) and (y_i') in s, then there is a least number k such that $x_k\neq x_k'$. From the formulas, $y_k\neq y_k'$ and thus $(y_i)\neq(y_i')$.

(2) Since, by formula, each coordinate function is continuous, then f must be continuous.

(3) To see that f^{-1} is continuous we consider solving for x_i in terms of $y_1, \cdots, y_i$. Clearly, for each i, x_i is a continuous function of $y_1, \cdots, y_i$. Also we know that for $(x_i)\in\bar{l}_2$, $\sum_{i=1}^{\infty} x_i^2=1$. Therefore f^{-1} must be continuous since convergence in the norm and convergence in each coordinate gives convergence in l_2.

(4) We wish to verify that $s\backslash f(\bar{l}_2)$ is a countable union of compact sets. Since l_2 is a separable complete metric space, we know that l_2 and thus $\bar{l}_2$ are absolute G_δ's. Letting $I^\infty=\prod_{i=1}^{\infty} I_i$ where I_i is the closed interval $[0, 1]$, we have $f(\bar{l}_2)\subset s\subset I^\infty$ and we know that $I^\infty\backslash f(\bar{l}_2)$ is a countable union of compact sets $(T_i)_{i\geq 1}$.

For each $j>0$, let $R_j=\{(y_i)\mid (y_i)\in s$ and, for each i, $1/2^j\leq y_i\leq 1-1/2^j\}$. Then R_j is a compact subset of s. Since $\{R_j\cap T_i\mid i, j>0\}$ is a countable union of compact sets, it suffices to show that for any point $(y_i)\in(s\backslash\bigcup_{j=1}^{\infty} R_j)$, there is a point $(x_i)\in\bar{l}_2$ for which $f((x_i))=(y_i)$.

Clearly, for each point $(y_i)\in s$, the formulas solved for the y_i's yield a particular sequence (x_i) with $\sum_{i=1}^{\infty} x_i^2\leq 1$. We note that (x_i) may have only finitely many nonzero coordinates but, in this case, $\sum_{i=1}^{\infty} x_i^2<1$.

Given $(y_i)\in(s\backslash\bigcup_{i=1}^{\infty} R_j)$ and any $\delta>0$ there exist an integer k and positive numbers ϵ_1 and ϵ_2 such that

$$0 < y_k < \epsilon_1 \quad \text{or} \quad 1 - \epsilon_1 < y_k < 1,$$

and this statement implies

$$\left(1 - \sum_{j=1}^{k-1} x_j^2\right)^{1/2} - \epsilon_2 < |x_k| < \left(1 - \sum_{j=1}^{k-1} x_j^2\right)^{1/2},$$

and this statement implies

$$0 < \left(1 - \sum_{j=1}^{k} x_j^2\right) < \delta \quad \text{and} \quad x_k \neq 0.$$

Therefore there is a point (x_i) of $\bar{l}_2$ such that $f((x_i)) = y_i$.

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