SMOOTH BANACH SPACES[1]

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1. **Introduction.** The purpose of this note is the study of twice differentiability of the norm in a real Banach space. We establish the various properties of the second derivative and obtain a polar characterization of twice differentiability of the norm. As a consequence of the various results a characterization of Hilbert spaces among Banach spaces which may be equipped with an equivalent twice differentiable norm is obtained.

2. **Notations and definitions.** Throughout this note E denotes a real Banach space with a Fréchet differentiable norm so that the spherical image map G on the unit sphere S of E into S^*, the unit sphere of E^* (the dual of E) is a function. For complete details and references about the first order differentiability of the norm in E in relation to the function G we refer to Cudia [1]. If $x \in S$ then E_x denotes the closed subspace $G(x)^{-1}(0)$.

DEFINITION. Let $(E, \|\cdot\|)$ be a Banach space. Then the norm is said to be twice differentiable at $x \neq 0$ if there exists a symmetric bilinear functional T_x on $E \times E$ such that

$$\|x + h\| = \|x\| + G(x)h + T_x(h, h) + \theta_x(h)$$

where $\theta_x(h)/\|h\|^2 \to 0$ as $\|h\| \to 0$ and $G(x)$ is the Gateux derivative of the norm at x. If the norm is twice differentiable at all members in S then the Banach space E is said to be twice Fréchet differentiable. The functional T_x may be identified as a bounded operator on E into E^* by the formula $\sigma(T_x)(y)z = T_x(y, z)$.

With the above notations we obtain the following theorems.

THEOREM 1. *If the norm of the Banach space E is twice differentiable at x then*

(i) *the norm is twice differentiable at all members λx, $\lambda \neq 0$ and $T_{\lambda x} = T_x/|\lambda|$.*

(ii) *$T_x(y, y) \geqq 0$ for all $y \in E$ and*

(iii) *the range of the operator $\sigma(T_x) \subseteq \{x\}^\perp$.*

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As a consequence of the above theorem we obtain the following polar characterization for the norm in E to be twice differentiable at a member $x \in S$.

THEOREM 2. *The norm in E is twice Frêchet differentiable at a member $x \in S$ if and only if the mapping ϕ defined on E_x into $\{x\}^{\perp}$ defined by setting $\phi(h) = \pi_x \circ G(x+h)$ where π_x is the projection on E^* into the closed subspace $Qx^{-1}(0)$, Qx being the canonical image of x in E^{**}, is Frechet differentiable.*

As a consequence of the above theorems we obtain the following isomorphism theorems.

THEOREM 3. *If the norm in E is twice differentiable at an element $x \neq 0$ and if the restriction of the operator $\sigma(T_x)$ to E_x is an isomorphism then E is isomorphic to a strictly convex Banach space. Further if $\inf_{y \in E_x \cap S} T_x(y, y) > 0$ then E is isomorphic to a Hilbert space.*

THEOREM 4. *If the Banach space E and its dual E^* are twice differentiable then E is isomorphic to a Hilbert space.*

The relationship between the twice differentiability of the norm and various notions of metric curvature [2] and the existence of free tangents [3] to a class of arcs on the unit sphere and the proofs of the above theorems will be appearing elsewhere.

REFERENCES

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