INTERPOLATION SPACES BY COMPLEX METHODS

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1. **Introduction.** We study some methods of constructing interpolation spaces "between" two Banach spaces. The methods extend the complex method introduced by Calderón [1] and Lions [2]. Our results generalize some of those of Calderón [3].

2. **Complex interpolation.** We shall call two Banach spaces X_0, X_1 compatible if they can be continuously embedded in a topological vector space V. We let X_0+X_1 denote the set of those elements $x \in V$ which can be written in the form

$$x = x^{(0)} + x^{(1)}, \tag{2.1}$$

where $x^{(j)} \in X_j$, $j=0, 1$. Set

$$\|x\|_{X_0+X_1} = \inf\{\|x^{(0)}\|_{X_0} + \|x^{(1)}\|_{X_1}\}, \tag{2.2}$$

where the infimum is taken over all pairs $x^{(j)} \in X_j$ satisfying (2.1). One easily checks that (2.2) gives a norm on X_0+X_1 when X_0 and X_1 are compatible. Moreover, when X_0+X_1 is equipped with this norm, it becomes a Banach space (cf. [3]).

Let $\mathcal{A}$ denote the set of complex valued functions of the complex variable $\zeta=\xi+i\eta$ which are continuous on bounded subsets of $\bar{\Omega}$, where Ω is the strip $0<\xi<1$ in the $\zeta=\xi+i\eta$ plane. Let $\mathcal{B}$ be the set of those functions in $\mathcal{A}$ which are holomorphic in Ω and nonvanishing in $\bar{\Omega}$. For $\rho \in \mathcal{A}$ the space $\mathcal{H}(X_0, X_1; \rho)$ will consist of those functions $f(\zeta)$ with values in X_0+X_1 such that (a) $f(\zeta)$ is continuous on bounded subsets of $\bar{\Omega}$, (b) $f(\zeta)$ is holomorphic in Ω, (c) $f(j+i\eta) \in X_j$, $j=0, 1$, η real, and

$$\|f(j + i\eta)\|_{X_j} \leqq \text{const } |\rho(j + i\eta)|. \tag{2.3}$$

Under the norm

$$\|f\|_{\mathcal{H}(X_0,X_1;\rho)} = \max_{j=0,1} \sup_{\eta} |\rho(j + i\eta)|^{-1}\|f(j + i\eta)\|_{X_j}, \tag{2.4}$$

$\mathcal{H}(X_0, X_1; \rho)$ becomes a Banach space.

Let T be a distribution with compact support in Ω (i.e., $T\in\mathcal{E}'(\Omega)$). For $\rho\in\mathcal{A}$ we let $X_{T,\rho} \equiv [X_0, X_1]_{T,\rho}$ denote the set of those $x\in X_0+X_1$ for which there is an $f\in\mathcal{H}(X_0, X_1; \rho)$ satisfying

$$x = T(f). \tag{2.5}$$

If we introduce the norm

$$\|x\|_{T,\rho} = \|x\|_{X_{T,\rho}} = \inf \|f\|_{\mathcal{H}(X_0,X_1;\rho)},$$

where the infimum is taken over all f which satisfy (2.5), then we have

PROPOSITION 2.1. *$X_{T,\rho}$ is a Banach space.*

Let $\mathbf{C}$ denote the (Banach space of) complex numbers.

PROPOSITION 2.2. *If $\rho, \sigma \in \mathcal{A}$ and $\omega \in \mathcal{H}(\mathbf{C}, \mathbf{C}; \sigma)$, then $X_{\omega T,\rho} \subseteq X_{T,\rho\sigma}$ with*

$$(2.6) \qquad \|x\|_{T,\rho\sigma} \leqq \|\omega\|_{\mathcal{H}(\mathbf{C},\mathbf{C};\sigma)} \|x\|_{\omega T,\rho}, \qquad x \in X_{\omega T,\rho}.$$

Set $X_T = X_{T,1}$. Then we have

PROPOSITION 2.3. *If $\rho \in \mathcal{B}$ then $X_{T,\rho} \equiv X_{T\rho}$ with the same norm.*

We let $X^{T,\rho} \equiv [X_0, X_1]^{T,\rho}$ designate the set of those $x \in X_0 + X_1$ for which there is an $f \in \mathcal{H}(X_0, X_1; \rho)$ such that

$$(2.7) \qquad fT = xT$$

in the sense of distributions. The norm in $X^{T,\rho}$ is given by

$$(2.8) \qquad \|x\|^{T,\rho} = \|x\|_{X^{T,\rho}} = \inf \|f\|_{\mathcal{H}(X_0,X_1;\rho)},$$

where the infimum is taken over all $f \in \mathcal{H}(X_0, X_1; \rho)$ satisfying (2.7). Set $X^T = X^{T,1}$.

PROPOSITION 2.4. *$X^{T,\rho}$ is a Banach space.*

PROPOSITION 2.5. *If $\rho, \sigma \in \mathcal{A}$ and $\omega \in C^\infty(\Omega)$, then $X^{T,\rho} \subseteq X^{\omega T,\rho}$ and*

$$\|x\|^{\omega T,\rho} \leqq \|x\|^{T,\rho}, \qquad x \in X^{T,\rho}.$$

If $\omega \neq 0$ on the support of T, the spaces are identical.

PROPOSITION 2.6. *If $\rho, \sigma \in \mathcal{B}$, $\rho T = \sigma T$, then $X_{T,\rho} \equiv X_{T,\sigma}$, $X^{T,\rho} \equiv X^{T,\sigma}$ with identical norms.*

Let $\mathcal{H}'(X_0, X_1; \rho)$ denote the space of (X_0+X_1)-valued functions $f(\zeta)$ on $\bar{\Omega}$ which are continuous on bounded subsets of $\bar{\Omega}$, holomorphic on Ω and such that $f(j+it_1) - f(j+it_2)$ is in X_j for all real $t_1, t_2, j = 0, 1$, and

$$(2.9) \quad \|f(j+it_2) - f(j+it_1)\|_{X_j} \leqq M \int_{t_1}^{t_2} |\rho(j+it)|\, dt, \qquad t_1 < t_2, j = 0,1.$$

The smallest constant M which works in (2.9) is the seminorm of f in $\mathcal{H}'(X_0, X_1; \rho)$. If one considers $\mathcal{H}'(X_0, X_1; \rho)$ modulo the constant functions, this becomes a norm and the resulting space is a Banach space. We say that $x \in X'_{T,p}$ if $x = T(f')$ for some $f \in \mathcal{H}'(X_0, X_1; \rho)$ and its norm is the infimum of the seminorms of all such f. Similarly, $x \in X'^{T,\rho}$ if $f'T = xT$ for some such f and its norm is defined correspondingly. If $f \in \mathcal{H}(X_0, X_1; \rho)$, then one checks easily that $\int_0^\zeta f(\zeta) d_\zeta$ is in $\mathcal{H}'(X_0, X_1; \rho)$ with seminorm not greater than the norm of f. Hence

$$X_{T,\rho} \subseteq X'_{T,\rho}, \qquad X^{T,\rho} \subseteq X'^{T,\rho} \tag{2.10}$$

with continuous injections.

PROPOSITION 2.7. *Propositions* 2.1–2.6 *hold true if each space is replaced by its primed counterpart.*

THEOREM 2.8. *Let Y_0, Y_1 be another pair of compatible Banach spaces and define $Y_{T,\rho}$, etc., in the same way. If L is a linear mapping of $X_0 + X_1$ into $Y_0 + Y_1$ which is bounded from X_j to Y_j, $j = 0, 1$, then it is a bounded mapping from $X_{T,\rho}$ to $Y_{T,\rho}$ and from $X^{T,\rho}$ to $Y^{T,\rho}$. The same holds true for the primed spaces.*

Let $Z_1, \cdots, Z_N$ be Banach spaces continuously imbedded in a topological vector space V. We let $\sum Y_n$ denote the Banach space consisting of those elements of V of the form $y = \sum y_n$, $y_n \in Y_n$, with norm given by $\|y\| = \inf \sum \|y_n\|_{Y_n}$. The space $\bigcap Y_n$ is the set of those y common to all the Y_n with norm

$$\|y\| = \max \|y\|_{Y_n}.$$

Set $\mathcal{H}(X_0, X_1) = \mathcal{H}(X_0, X_1, 1)$.

PROPOSITION 2.9. *Assume that there are distributions $T_1, \cdots, T_N$ and functions $\omega_1, \cdots, \omega_N$; $\tau_1, \cdots, \tau_N$ in $\mathcal{H}(\mathbf{C}, \mathbf{C})$ such that*

$$T = \sum \omega_n T_n, \qquad T_n = \tau_n T. \tag{2.11}$$

Then $X_{T,\rho} \equiv \sum X_{T_n,\rho}$, $X^{T,\rho} \equiv \bigcap X^{T_n,\rho}$, with equivalent norms. The same relationship holds for the primed spaces.

PROPOSITION 2.10. *If $\rho, \sigma \in \mathfrak{A}$ satisfy*

$$\left| \frac{\rho(j + it)}{\sigma(j + it)} \right| \leqq M, \qquad j = 0, 1, t \text{ real}$$

then $X_{T,\rho} \subseteq X_{T,\sigma}$, $X^{T,\rho} \subseteq X^{T,\sigma}$ and

$$\|x\|_{T,\sigma} \leqq M \|x\|_{T,\rho}, \qquad \|x\|^{T,\sigma} \leqq M \|x\|^{T,\rho}.$$

The same holds for the primed spaces.

We now assume that the support of T consists of a finite number of points $z_1, \cdots, z_N$ in Ω. When acting on holomorphic functions T can be written in the form

$$T = \sum_{k=1}^{N} \sum_{l=0}^{m_k} a_{kl}\delta^{(l)}(z_k). \tag{2.12}$$

In this case we have

THEOREM 2.11. *If $\rho \in \mathfrak{A}$ and $\omega \in C^\infty(\Omega)$, then $X_{\omega T,\rho} \subseteq X_{T,\rho}$ with continuous injection. If $\omega \neq 0$ on the support of T, then the spaces are equivalent. Similar statements are true for the primed spaces.*

COROLLARY 2.12. *If $\rho \in \mathfrak{B}$, then $X_{T,\rho} \equiv X_T$.*

THEOREM 2.13. *If $X_0 \cap X_1$ is dense in both X_0 and X_1, then the dual of X_T is isomorphic to $[X_0^*, X_1^*]'^T$; that of X^T to $[X_0^*, X_1^*]'_T$.*

THEOREM 2.14. *If either X_0 or X_1 is reflexive, then $X'_T \equiv X_T$ and $X'^T \equiv X^T$. Moreover, both of these spaces are reflexive when $X_0 \cap X_1$ is dense in both X_0 and X_1.*

If $a_{k,m_k} \neq 0$, we shall say that the distribution (2.12) is of order m_k at z_k. We shall say that T_0 is contained in T and write $T_0 \subseteq T$ if the support of T_0 is contained in that of T and it is not of greater order than T at any point of its support.

THEOREM 2.15. *If $T_0 \subseteq T$, then $X_{T_0,\rho} \subseteq X_{T,\rho}$ and $X^{T_0,\rho} \supseteq X^{T,\rho}$ with continuous inclusions. The same holds for the primed spaces.*

THEOREM 2.16. *If $T = \sum T_n$ and each $T_n \subseteq T$, then*

$$X_{T,\rho} \equiv \sum X_{T_n,\rho}, \qquad X^{T,\rho} \equiv \bigcap X^{T_n,\rho}.$$

A similar statement is true for the primed spaces.

THEOREM 2.17. *$X_0 \cap X_1$ is dense in X_T. If $X_0 \cap X_1$ is dense in both X_0 and X_1, it is dense in X^T.*

THEOREM 2.18. *If $X_0 \cap X_1$ is dense in both X_0 and X_1, then for $0 \leqq \theta_1 < \theta_2 \leqq 1$*

$$[X_{\theta_1}, X_{\theta_2}]_{\delta(\theta)} \equiv X_{\theta_1 + \theta(\theta_2 - \theta_1)}$$

with equivalent norms, where $X_\theta = X_{\delta(\theta)}$.

3. **Some examples.** We consider distributions $u(x)$ on E^n, $x = (x_1, \cdots, x_n)$. Let $\mathfrak{F}u(\xi)$ denote the Fourier transform of $u(x)$, $\xi = (\xi_1, \cdots, \xi_n)$. For a positive function $\lambda(\xi)$ we let $B^{\lambda,p}$ denote the set of those $u(x)$ such that $\lambda\mathfrak{F}u \in L^p(E^n)$, $1 \leqq p \leqq \infty$. The norm of $u(x)$ in $B^{\lambda,p}$ is the L^p norm of $\lambda\mathfrak{F}u$. These spaces were studied by Hörmander [4] when the function $\lambda(\xi)$ satisfies certain conditions.

Let $\lambda_0(\xi)$, $\lambda_1(\xi)$ be functions satisfying $\lambda_1(\xi) \geqq \lambda_0(\xi) > 0$. We assume that the distribution T is of the form (2.12). Set $\gamma = \lambda_1/\lambda_0$, $s_k = \operatorname{Re} z_k$,

$$\alpha = \sum \gamma^{s_k}(1 + \log \gamma)^{m_k}, \qquad \beta = \sum \gamma^{-s_k}(1 + \log \gamma)^{m_k},$$

$\lambda_2 = \lambda_0 \alpha$, $\lambda_3 = \lambda_0/\beta$. We also write

$$B_T = [B^{\lambda_0,p}, B^{\lambda_1,p}]_T, \qquad B^T = [B^{\lambda_0,p}, B^{\lambda_1,p}]^T.$$

THEOREM 3.1. *$B_T \equiv B^{\lambda_3,p}$, $B^T \equiv B^{\lambda_2,p}$ with equivalent norms.*

The proofs of Theorem 3.1 can be made to depend upon the following lemmas.

LEMMA 3.2. *$u \in B_T$ if and only if there is a $g \in \mathcal{H}(L^p, L^p)$ such that*

$$\mathfrak{F}u = \frac{1}{\lambda_0} T(\gamma^{-\zeta} g).$$

LEMMA 3.3. *Let $z_1, \cdots, z_N$ be fixed points in Ω and let $\{\nu_{kl}\}$, $1 \leqq k \leqq N$, $0 \leqq l \leqq m$, be given complex numbers. Then one can find a function $\omega \in \mathcal{H}(\mathbf{C}, \mathbf{C})$ satisfying*

$$\omega^{(l)}(z_k) = \nu_{kl}, \qquad 1 \leqq k \leqq N, \ 0 \leqq l \leqq m,$$

$$\|\omega\|_{\mathcal{H}(\mathbf{C},\mathbf{C})} \leqq K \sum |\nu_{kl}|,$$

where the constant K depends only on the z_k and m.

COROLLARY 3.4. *If $T_0 \subseteq T$, there is a $\tau \in \mathcal{H}(\mathbf{C}, \mathbf{C})$ such that $T_0 = \tau T$.*

The second family of function spaces we shall consider is closely related to the first. For positive $\lambda(\xi)$ we let $H^{\lambda,p}$ denote the set of those distributions $u(x)$ such that $\mathfrak{F}^{-1}\lambda\mathfrak{F}u \in L^p(E^n)$, where $\mathfrak{F}^{-1}$ denotes the inverse Fourier transform. When $\lambda(\xi)$ is of the form

$$\lambda(\xi) = (1 + |\xi|^2)^a, \quad a \text{ real}, \tag{4.6}$$

these spaces were studied by Calderón [5], Aronszajn, Mulla, Szeptycki [6], Lions, Magenes [7] and others.

In order to obtain the counterpart of Theorem 3.1 for the spaces $H^{\lambda,p}$ we shall make further restrictions. Specifically we shall assume that $1 < p < \infty$ and that the functions

$$\gamma^{it}, \quad t \text{ real},$$

$$\gamma^{z_k}(\log \gamma)^m/\alpha, \qquad \gamma^{-z_k}(\log \gamma)^m/\beta, \qquad 1 \leqq k \leqq N, \ m \leqq m_k$$

all belong to the space M_p of multipliers in L^p. Moreover the norms of γ^{it} in M_p are to be uniformly bounded. By employing Mihlin's theorem [8], one can show easily that these assumptions are implied by

$$|\xi|^{|\mu|}|D^\mu\gamma(\xi)| \leqq \text{const } |\gamma(\xi)|, \quad |\mu| \leqq n,$$

which in turn holds when λ_0 and λ_1 are of the form (4.6).

Under the above assumptions we can state

THEOREM 3.5. *$H_T \equiv H^{\lambda_s,p}$, $H^T \equiv H^{\lambda_s,p}$ with equivalent norms, where*

$$H_T = [H^{\lambda_0,p}, H^{\lambda_1,p}]_T, \qquad H^T = [H^{\lambda_0,p}, H^{\lambda_1,p}]^T.$$

The proof of Theorem 3.5 follows that of Theorem 3.1 very closely. In place of Lemma 4.2 we have

LEMMA 3.6. *$u \in H_T$ if and only if there is a $g \in \mathcal{H}(L^p, L^p)$ such that*

$$\mathfrak{F}u = \frac{1}{\lambda_0} T(\gamma^{-\zeta}\mathfrak{F}g).$$

4. **Functions of an operator.** Let X be a Banach space and A a closed linear operator in X with dense domain $D(A)$. We assume that the resolvent of A contains the negative real axis and that

$$\|(\lambda + A)^{-1}\| \leqq M(1 + \lambda)^{-1}, \qquad \lambda > 0. \tag{4.1}$$

Let $\phi(\zeta, \lambda)$ be a complex valued function defined for $\zeta \in \bar{\Omega}$ and $0 < \lambda < \infty$ and satisfying the following conditions: (a) for each $\zeta \in \bar{\Omega}$, $\phi(\zeta, \lambda)$ is measurable (as a function of λ) and

$$\int_0^\infty |\phi(\zeta, \lambda)| (1 + \lambda)^{-1} d\lambda \tag{4.2}$$

is finite, (b) considered as a function of ζ, $\phi(\zeta, \lambda)$ is continuous in $\bar{\Omega}$ with respect to the norm (4.2), (c) $\phi(\zeta, \lambda)$ is an analytic function of ζ in Ω with respect to the same norm, (d) there is a $\rho \in \mathcal{B}$ such that

$$\int_0^\infty |\phi(j + i\eta, \lambda)| (1 + \lambda)^{j-1} d\lambda \leqq |\rho(j + i\eta)|, \qquad \eta \text{ real}, j = 0, 1.$$

Under the above assumptions we define the following family of operators

$$\psi(\zeta, A) = \int_0^\infty \phi(\zeta, \lambda)(\lambda + A)^{-1} d\lambda, \qquad \zeta \in \bar{\Omega}.$$

PROPOSITION 4.1. *For real η the operator $\psi(1+i\eta, A)$ maps X into $D(A)$ and*

$$\|A\psi(1 + i\eta, A)\| \leqq (M + 1) |\rho(1 + i\eta)|. \tag{4.5}$$

Now we consider $D(A)$ as a Banach space contained in X with norm $\|x\|+\|Ax\|$. We set

$$\psi^{(m)}(\zeta, A) = \frac{d^m}{d\zeta^m}\psi(\zeta, A) = \int_0^\infty \frac{\partial^m \phi(\zeta, \lambda)}{\partial \zeta^m} (\lambda + A)^{-1}\, d\lambda,$$

$$m = 0, 1, 2, \cdots,$$

$$\psi_T(A) = T[\psi(\zeta, A)] = \int_0^\infty T[\phi(\zeta, \lambda)](\lambda + A)^{-1} d\lambda, \quad T \in \mathcal{E}'(\Omega).$$

THEOREM 4.2. *The operator* $\psi_T(A)$ *maps* X *boundedly into*

$$X_{T,\rho} = [X, D(A)]_{T,\rho}$$

with norm $\leqq M+2$. *In particular* $\psi^{(m)}(\theta, A)$ *maps* X *into*

$$X_{\theta,\rho}^{(m)} = [X, D(A)]_{\delta^{(m)}(\theta),\rho}.$$

LEMMA 4.3. *For real* η *the operator* $\psi(i\eta, A)$ *maps* $D(A)$ *into itself with norm* $\leqq M|\rho(i\eta)|$.

THEOREM 4.4. *For* $0 \leqq \theta+\theta_0 \leqq 1$, $\psi(\theta, A)$ *maps* X_{θ_0} *boundedly into* $X_{\theta_0+\theta}$ *with norm* $\leqq M^{1-\theta}(M+2)^\theta |\rho(\theta)|$.

Let Y be a Banach space and let B be an operator defined in Y with the same properties as A. In particular, we assume

$$\|(\lambda + B)^{-1}\| \leqq M(1 + \lambda)^{-1}, \qquad \lambda > 0.$$

Let L be a linear operator which maps X into Y in such a way that $D(A)$ maps into $D(B)$. Assume that

$$\|Lx\| \leqq K_0\|x\|, \qquad x \in X,$$
$$\|BLx\| \leqq K_1\|Ax\|, \qquad x \in D(A).$$

Then we have

THEOREM 4.5. *For* $0 \leqq \theta \leqq 1$ *the operator* $\psi(1-\theta, B)L\psi(\theta, A)$ *maps* X *into* $D(B)$ *and*

$$\|B\psi(1 - \theta, B)L\psi(\theta, A)\| \leqq M(M + 2)K_0^{1-\theta}K_1^{\theta} |\rho(\theta)\rho(1 - \theta)|.$$

Next consider the operators A^θ defined by

$$A^{-\theta} = \frac{\sin \pi\theta}{\pi}\int_0^\infty \lambda^{-\theta}(\lambda + A)^{-1}\, d\lambda, \qquad 0 < \theta < 1,$$

as given by Kato [11]. If we set

$$\phi(\zeta, \lambda) = \lambda^{-\zeta}[(\log \lambda)^2+\pi^2]^{-2}\{[(\log \lambda)^2-\pi^2] \sin \pi\zeta + 2\pi \log \lambda \cos \pi\zeta\},$$

then the corresponding operator $\psi(\zeta, A)$ satisfies all of the requirements above with $p=1$. Moreover

$$A^{-\theta} = \psi^{(2)}(\theta, A).$$

Hence we have by Theorem 4.2

PROPOSITION 4.6. *The domain of A^{θ} is contained in $X_{\theta}^{(2)}$.*

We also have

PROPOSITION 4.7. *The operator $A^{-\theta}$ maps X_{θ_0} boundedly into $X_{\theta_0+\theta-\epsilon}$ for every $\epsilon>0$, and the operator A^{θ} maps X_{θ_0} boundedly into $X_{\theta_0-\theta-\epsilon}$ for each $\epsilon>0$. In particular, the domain of A^{θ} is contained in $X_{\theta-\epsilon}$ for each $\epsilon>0$.*

Our proof of Proposition 4.7 rests on

LEMMA 4.8. *For $x \in X_{\theta_0}$ we have*

$$\|(\lambda + A)^{-1}x\|_{\delta(\theta_0+\theta)} \leqq M^{1-\theta}(M+2)^{\theta}(1+\lambda)^{\theta-1}\|x\|_{\delta(\theta_0)},$$
$$\|A(\lambda + A)^{-1}x\|_{\delta(\theta_0-\theta)} \leqq (M+1)^{1-\theta}M^{\theta}(1+\lambda)^{-\theta}\|x\|_{\delta(\theta_0)}.$$

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