SELF-EQUIVALENCES OF $(n-1)$-CONNECTED $2n$-MANIFOLDS[1]

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1. Introduction and statement of main results. All spaces have basepoints, and all maps of spaces are basepoint-preserving. A self-equivalence of a space X is a homotopy class of homotopy equivalences $X \to X$. Map-composition induces an operation on the set of self-equivalences of X, making it into a group, $\mathcal{E}(X)$.

Arkowitz and Curjel [1] and Weishu Shih [7] have obtained certain general results about $\mathcal{E}(X)$ by studying the Postnikov decomposition of X. More recently P. Olum [5] presented an explicit computation of $\mathcal{E}(X)$ in the case that X is a pseudo-projective plane.

Our results concern the structure of $\mathcal{E}(X)$ in the case that X is a closed, compact, oriented, C^∞, $(n-1)$-connected $2n$-manifold, $n \geqq 2$. We place these restrictions on X throughout the rest of this paper. Our methods are dual to those of [1] and [7] in the sense that we proceed by examining a cell-decomposition of X.

A word about notation: X_n is the n-skeleton of X in some fixed, minimal CW-decomposition of X, SX_n is its suspension, and $\pi(SX_n, X)$ is the group of homotopy classes of maps $SX_n \to X$.

THEOREM 1. *There is an exact sequence,*

$$\pi(SX_n, X) \xrightarrow{(Sb)^* + \bar{\psi}} \pi_{2n}(X) \xrightarrow{\rho} \mathcal{E}(X) \xrightarrow{R} \mathcal{E}(X_n),$$

the homomorphisms of which will be described in §2.

It is easy to show that $\pi_{2n}(X)$ is finite.

COROLLARY TO THEOREM 1. *Kernel R is finite.*

X_n is a one-point union of (at least two) n-spheres, so that $H_n(X_n) = H_n(X)$ is finitely generated free abelian. Moreover, it is easy to show that the homology functor H_n takes $\mathcal{E}(X_n)$ isomorphically onto the group of automorphisms of $H_n(X)$. We call this automorphism group $\mathrm{Aut}(H_n(X))$.

Let $\mu: H_n(X) \otimes H_n(X) \to Z$ be the integral bilinear form determined by the intersection pairing on $H_n(X)$. Wall [8] shows that μ, together with a certain function $H_n(X) \to \pi_{2n-1}(S^n)$, completely deter-

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mines the homotopy type of X. For algebraic convenience, we modify this function slightly, obtaining a homomorphism c on $H_n(X)$, which together with μ also determines the homotopy type of X. We do not define c here.

Let $\mathrm{Aut}(\mu, c)$ be the subgroup of $\mathrm{Aut}(H_n(X))$ consisting of all automorphisms that preserve c and that, up to sign, preserve μ.

THEOREM 2. *The functor H_n maps image R isomorphically onto $Aut(\mu, c)$.*

THEOREM 3. $\mathrm{Aut}(\mu, c)$ *is finitely generated. If n is even and μ is a definite quadratic form, or if n is even and μ has rank two and index zero, then* $\mathrm{Aut}(\mu, c)$ *is finite. Otherwise,* $\mathrm{Aut}(\mu, c)$ *is infinite.*

Combining Theorems 2 and 3 with the fact that kernel R is finite, we obtain the following:

COROLLARY TO THEOREM 3. *Theorem 3 holds for $\mathcal{E}(X)$ in place of* $\mathrm{Aut}(\mu, c)$.

Let $\mathfrak{D}(X)$ be the subgroup of $\mathcal{E}(X)$ consisting of all classes represented by diffeomorphisms $X \to X$.

THEOREM 4. *Suppose that $n \equiv 2$* (mod 4), *$n \neq 2$. There is a number k, depending only on n and on* rank $(H_n(X))$, *such that the index of $\mathfrak{D}(X)$ in $\mathcal{E}(X)$ is less than k.*

COROLLARY TO THEOREM 4. *Under the above restriction on n, Theorem 3 holds for $\mathfrak{D}(X)$ in place of* $\mathrm{Aut}(\mu, c)$.

EXAMPLES.

(a) Let CP^n be complex projective n-space. Using the exact sequence of Theorem 1, together with well-known facts about the homotopy type of CP^2, it is easy to calculate that $\mathcal{E}(CP^2) \cong Z_2$.

Indeed, an easy but unrelated argument shows that $\mathcal{E}(CP^n) \cong Z_2$, for all $n \geqq 1$.

(b) Let KP^n be quaternion projective n-space. Using Theorem 1 again, together with certain accessible but less well-known facts about the homotopy type of KP^2, one may calculate that $\mathcal{E}(KP^2) \cong Z_2$.

In contrast to the above example, however, image R here is trivial. This implies:

PROPOSITION 1. *Every homotopy equivalence f: $KP^n \to KP^n$, $n \geqq 2$, induces the identity automorphism of cohomology.*

(c) We determine $\mathcal{E}(S_1^n \times S_2^n)$, $n \geqq 2$. In this case, X_n is the one-point union $S_1^n \vee S_2^n$. We need some notation:

i is the homotopy class of the inclusion $S_1^n \vee S_2^n \to S_1^n \times S_2^n$;
e_l is the element of $\pi_n(S_1^n \vee S_2^n)$ represented by the inclusion of S^n onto S_l^n, $l=1, 2$;
x is the homotopy class of the Hopf map $S^3 \to S^2$, $S^{n-2}x$ its $(n-2)$-fold suspension;
ι_n is the homotopy class of the identity map $S^n \to S^n$;
$[\alpha, \beta]$ is the Whitehead product of homotopy classes α and β;
Δ_8 is the dihedral group of order eight, a group on two generators a and b satisfying $a^4 = b^2 = ab^{-1}ab = 1$;
Sym is the group of integral 2×2 matrices generated by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \text{(cf., [3]);}$$

Δ will be the image of the homomorphism $(Sb)^* + \bar{\psi}$ of Theorem 1.

PROPOSITION 2. (i) *Δ is trivial if $n=2$, 6 or $n \equiv 3 \pmod 4$. Otherwise $\Delta \cong Z_2 \oplus Z_2$ and is generated by*

$$i \circ e_1 \circ [S^{n-2}\, x, \iota_n] \quad \textit{and} \quad i \circ e_2 \circ [S^{n-2}\, x, \iota_n].$$

(ii) *If n is odd,* image $R \cong \mathrm{Sym}$, *whereas if n is even,* image $R \cong \Delta_8$.
(iii) *The following sequence is split-exact:*

$$0 \to \pi_{2n}(S_1^n \times S_2^n)/\Delta \xrightarrow{\rho} \mathcal{E}(S_1^n \times S_2^n) \xrightarrow{R} \text{image } R \to 0.$$

The action of Sym or Δ_8 on $\pi_{2n}(S_1^n \times S_2^n/\Delta$ can be computed explicitly, so that in the range of values of n for which $\pi_{2n}(S^n)$ is known, $n \geqq 2$, $\mathcal{E}(S_1^n \times S_2^n)$ can be completely determined.

(d) We present an example of a $(4k-1)$-connected $8k$-manifold M, $k \geqq 2$, such that the index of $\mathfrak{D}(M)$ in $\mathcal{E}(M)$ is $\geqq 8$.

Choose any of the manifolds M constructed in [4] such that (i) M is homotopically equivalent to $S_1^{4k} \times S_2^{4k}$; (ii) the Pontrjagin class $p_k(M) = ae_1^* + be_2^*$, where $0 \neq a \neq \pm b \neq 0$ and e_l^* is the generator of $H^{4k}(M)$ corresponding, via the given homotopy equivalence, Poincaré duality, and the Hurewicz isomorphism, to the homotopy class e_l described in (c), $l=1, 2$.

It is easy to show that, of all the members of image $R \cong \Delta_8$, only the identity induces an automorphism of cohomology that keeps $p_k(M)$ fixed. Since diffeomorphisms induce cohomology isomorphisms that keep Pontrjagin classes fixed, $R(\mathfrak{D}(M))$ is trivial, from which the result follows.

2. Description of the homomorphisms and of the proof of Theorem 1.

DEFINITION OF R: $\mathcal{E}(X)\to\mathcal{E}(X_n)$. $R(f)$ is the homotopy class of the restriction to X_n of any cellular representative of f. J. H. C. Whitehead's Cellular Approximation Theorem implies that R is well-defined.

DEFINITION of ρ: $\pi_{2n}(X)\to\mathcal{E}(X)$. As a CW-complex, $X=X_n\cup e^{2n}$, where the cell e^{2n} is attached to X_n by a map b: $S^{2n-1}\to X_n$. Therefore, we may identify X with the reduced mapping cone of b. Pinching together all points halfway up the cone, we obtain $S^{2n}\vee X$ and a projection π: $X\to S^{2n}\vee X$. Given any a: $S^2{}_n\to X$, it determines a map $(a\vee 1)\circ\pi$: $X\to X$, where 1 is the identity map of X. Passing to homotopy classes, the association $a\to(a\vee 1)\circ\pi$ determines the homomorphism ρ (cf. [1], and [2, p. 179]).

DEFINITION OF $(Sb)^*$: $[SX_n, X]\to\pi_{2n}(X)$. b: $S^{2n-1}\to X_n$ is the attaching map of e^{2n}, as above, Sb is its suspension, and $(Sb)^*$ is determined by right composition with Sb.

DEFINITION OF Ψ: $[SX_n, X]\to\pi_{2n}(X)$. We introduce notation analogous to that of example (c), above:

i is the homotopy class of the inclusion $X_n\subset X$;

e_k is the homotopy class of the inclusion of S^n onto the kth sphere of the one-point union of n-spheres X_n;

$S\alpha$ is the suspension of α, and $[\alpha, \beta]$ is the Whitehead product of α, β;

(Γ_{lk}) is the unimodular matrix determined by the cup product of $H^*(X)$ with respect to the basis of $H^n(X)=\mathrm{Hom}(H_n(X), Z)$ dual to $\{e_1, e_2, \cdots\}\subset\pi_n(X_n)=H_n(X_n)=H_n(X)$. Then, we define Ψ by

$$\Psi(x) = \sum_{l,k}\Gamma_{lk}[x\circ Se_l, i\circ e_k].$$

Ψ arises roughly because of the failure of right composition with b to determine a homomorphism $\pi(X_n, X)\to\pi_{2n-1}(X)$.

REMARKS ON THE PROOF OF THEOREM 1. The proof of Theorem 1 is an easy obstruction-theoretic exercise until one gets to proving exactness at $\pi_{2n}(X)$. At this point it is necessary to characterize a certain obstruction set (see [2, p. 185]). It is not at all difficult to show that this set is *some* homomorphic image of $[SX_n, X]$. The difficulty lies in showing that the homomorphism is $(Sb)^*+\Psi$.

The arguments in this proof can be generalized. However, in general, image R will not have so simple a description as that supplied by Theorem 2.

BIBLIOGRAPHY

1. M. Arkowitz and C. Curjel, *The group of homotopy equivalences of a space*, Bull. Amer. Math. Soc. **70** (1964), 293–296.
2. Sze-tsen Hu, *Homotopy theory*, Academic Press, New York, 1959.
3. L. K. Hua and I. Reiner, *On the generators of the symplectic modular group*, Trans. Amer. Math. Soc. **65** (1949), 415–426.
4. P. J. Kahn, *Characteristic numbers and oriented homotopy type*, Topology **3** (1965), 81–95.
5. P. Olum, *Self-equivalences of pseudo-projective planes*, Topology **4** (1965), 109–127.
6. D. Puppe, *Homotopiemengen und ihre induzierten Abbildungen*. I, Math. Z. **69** (1958), 299–344.
7. Weishu Shih, *On the group* $\mathcal{E}[X]$ *on homotopy equivalence maps*, Bull. Amer. Math. Soc. **70** (1964), 361–365.
8. C. T. C. Wall, *Classification of* (n−1)*-connected* 2n*-manifolds*, Ann. of Math. (2) **75** (1962), 163–182.

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THE SOLUTION BY ITERATION OF LINEAR FUNCTIONAL EQUATIONS IN BANACH SPACES[1]

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Let X be a Banach space (real or complex), T a bounded linear operator from X to X. We are concerned with the solution of the equation

$$u - Tu = f, \tag{1}$$

by the iteration process of Picard-Poincaré-Neumann,

$$x_{n+1} = Tx_n + f \quad (x_0 \text{ given}), \tag{2}$$

i.e. with the convergence of the sequence

$$x_n = T^n x_0 + (f + Tf + \cdots + T^{n-1}f).$$

By an earlier result of the first-named author (Browder [2]), if X is reflexive, a solution u for the equation (1) will exist for a given element f of X and an operator T which is *asymptotically bounded* (i.e. $\|T^k\| \leqq M$ for some $M>0$ and all $k \geqq 1$) if and only if the sequence $\{x_n\}$ is bounded for any fixed x_0. Our object in the present paper is to

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